Sand Stat, 4.aflevering

ldg790 - Christian B. Gustafson

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Opgave U44.A:

1

For at vise tætheden for (X, Y) gør jeg brug af den kontinuerte fordeling med sandsynlighedstætheden

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} . x \in R,$$

der kaldes standard normalfordelingen.

$$p(x,y) = \varphi(x)\varphi(y) = \frac{1}{\sqrt{2\pi}} exp(-x^2/2) \cdot \frac{1}{\sqrt{2\pi}} exp(-y^2/2)$$

$$= \frac{1}{\sqrt{2\pi}} exp(-(x^2+y^2)/2) = \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}(x^2+y^2)$$

2

For at vise tæthen for Z benytter jeg korollar 6.3.2, hvor x=2 og y=z-w

$$q(z) = \int_{-\infty}^{\infty} p_1(x)p_2(z-x)dx = \int_{-\infty}^{\infty} p(w, z-w)dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(-\frac{1}{2}(w^2 + (z-w)^2))dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(-\frac{1}{2}((z-w)^2 + w^2))dw$$

3

For at vise q(z) vil jeg gøre følgende:

$$q(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(-\frac{1}{2}((z-w)^2 + w^2))dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(-\frac{1}{2}(z^2 - 2zw + 2w^2))dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(-\frac{1}{2}z^2 - zw + w^2)dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(-\frac{1}{4}z^2 - \frac{1}{4}z^2 + zw - w^2)dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(-\frac{1}{4}z^2 + \frac{1}{4}z^2 - \frac{1}{2}(z^2 - 2zw + 2w^2))dw$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(-\frac{1}{4}z^2) exp(\frac{1}{4}z^2 - \frac{1}{2}(z^2 - 2zw + 2w^2))dw$$

$$= exp(-\frac{1}{4}z^2) \int_{-\infty}^{\infty} \frac{1}{2\pi} exp(\frac{1}{4}z^2 - \frac{1}{2}(z^2 - 2zw + 2w^2))dw$$

$$= exp(-\frac{1}{4}z^2) \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} \cdot \frac{1}{\sqrt{\pi}} exp(\frac{1}{4}z^2 - \frac{1}{2}(z^2 - 2zw + 2w^2))dw$$

$$= \frac{1}{\sqrt{4\pi}} exp(-\frac{1}{4}z^2) \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} exp(\frac{1}{4}z^2 - \frac{1}{2}(z^2 - 2zw + 2w^2))dw$$

4

Tæthedsfunktionen for $N(\mu, \sigma^2)$ fordeling er:

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{(z-\mu)^2}{2\sigma^2})$$

Dvs. tætheden for $N(0, 2)$ fordelingen er følgende:

$$p(z) = \frac{1}{\sqrt{2\pi \cdot 2}} exp(-\frac{(z-0)^2}{2 \cdot 2})$$

$$p(z) = \frac{1}{\sqrt{4\pi}} exp(-\frac{1}{4}z^2)$$

5

For at vise dette vil gøre følgende:

$$m = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} exp(\frac{1}{4}z^2 - \frac{1}{2}(z^2 - 2zw + 2w^2))dw$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} exp(\frac{1}{4}z^2 - \frac{1}{4}z^2 - \frac{1}{4}z^2 + zw - w^2))dw$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} exp(\frac{1}{4}z^2 + zw - w^2))dw$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} exp(\frac{1}{2}(\sqrt{2w} - \frac{1}{\sqrt{2}}z)^2)dw$$

For at vise det næste vil jeg substituere $v = \sqrt{2w} - \frac{1}{2}z$:

$$\frac{dv}{dw} = \sqrt{2} \Rightarrow dw = \frac{1}{\sqrt{2}}dv$$

$$m = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} exp(-\frac{1}{2}v^2) \cdot \frac{1}{\sqrt{2}}dv = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}v^2)dv$$

Fordi at $p(v) = \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}v^2)dv$ kan jeg bruge 5.1.2, som fortæller os at:

En funktion p fra et interval $I \subseteq R \in [0, \infty)$ der opfylder at:

$$\int_1 p(v)dv = \int_{-\infty}^{\infty} 1(x \in I)p(v)dx = 1$$

kaldes en ssh-tæthed på I. Dvs. at

$$m = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}v^2)dv = 1$$

6

Eftersom at $X_1, ..., X_m$ er uafhængige normalfordelte, så er summen af dem normalfordelte, dvs.

$$Z \approx N(0M, 1M) = N(0, M)$$

Deraf ved vi for regnereglerne for middelværdi og varians at:

$$aX \approx N(aE(X), a^2Var(X)),$$

dvs.

$$\frac{Z}{M} = \frac{1}{M}\sum_{i=1}^{M} X_i \approx N(0 \cdot \frac{1}{M}, M \cdot (\frac{1}{M})^2) = N(0, 1\frac{1}{M})$$

Så kan vi se at når $M \to 0$ vil $Var\frac{Z}{M} \to 0$