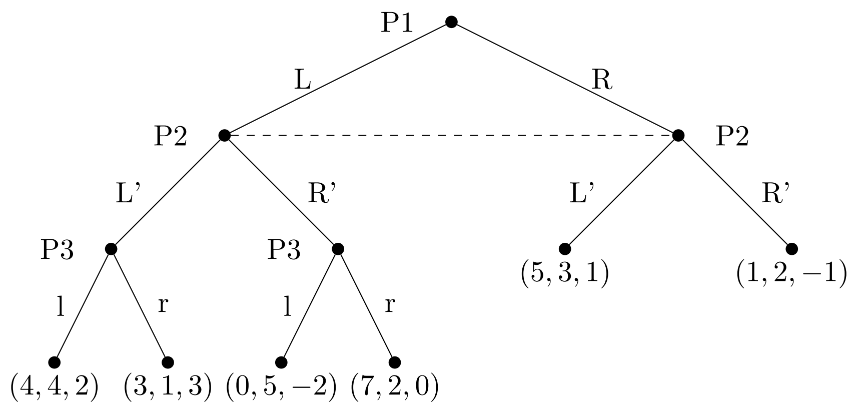
Microeconomics III
Fall 2021

Assignment 2 Solutions

To pass, you need to get at least half of the points. All three questions are of equal value.

1. Consider the figure below. The first payoff is that of player 1, the second that of player 2, the third that of player 3.



(a) Is this a game of perfect or imperfect information? How many subgames does the game have? What are the strategy sets?

Solution: This is a game of imperfect information, since P2 cannot observe P1's action in stage 1.

There are 2 proper subgames, one starting at each of P3's decision nodes.

The strategy sets are:

$$S_1 = \{L, R\}$$

$$S_2 = \{L^{'}, R^{'}\}$$

$$S_3 = \{ll, lr, rl, rr\}$$

(b) Find the set of pure-strategy SPE of the game.

Solution: In stage 3, P3 will play r in both subgames. Stage 1 and 2 can then be represented by the following bi-matrix

		P2	
		L'	R'
P1	L	3,1	7,2
	R	5,3	1,2

Both $(L, R^{'})$ and $(R, L^{'})$ are NE of the reduced game resulting in the following set of SPE:

$$SPE : \{(L, R^{'}, rr), (R, L^{'}, rr)\}$$

(c) Now suppose we split the information set such that player 2 can observe player 1's choice. What is the set of pure-strategy SPE?

Solution:

When we split the information set, the game becomes a game of perfect information.

There are now 4 proper subgames. On top of the ones specified in $1a$, there is one starting at each of P2's decision nodes (including all following decision nodes and terminal nodes) .

The strategy sets are:

$$S_1 = \{L, R\}$$

$$S_2 = \{L^{'}L^{'}, L^{'}R^{'}, R^{'}L^{'}, R^{'}R^{'}\}$$

$$S_3 = \{ll, lr, rl, rr\}$$

In stage 3, P3 will still play r in both subgames. In stage 2, P2 will play $R^{'}$ at their left decision node and $L^{'}$ at their right decision node. In stage 1, P1 will play L.

The set of SPE is then:

$$SPE : (L, R^{'}L^{'}, rr)$$

(d) Does player 2 gain from being able to observe player 1's choice? Why/why not?

Solution: P2 does not gain from being able to observe P1's choice. In the game with imperfect information, 2 SPE existed with payoffs for P2 of 3 and 2 respectively. In the perfect information game, only one SPE exist with payoff 2 for P2, i.e. letting P2 observe P1's choice eliminates the possibility for P2 to get the higher payoff. The reason is, that when we allow P2 to observe P1's action, we give P1 commitment power. P1 can now steer the game towards the SPE that is most profitable for them, which is simultaneously the least profitable of the SPE for P2.

2. Consider the following normal form game:

	a	b	c
A	$0,0$	$4,4$	$6,0$
B	$1,2$	$0,0$	$2,2$
C	$0,6$	$2,2$	$5,5$

(a) Find all mixed and pure NE for the static game and calculate the corresponding NE payoffs for both players. (Notice that the question was later modified to be such that 3 NE is enough!)

Solution: The pure strategy NE can be found by plotting in the best responses in the bi-matrix. There are two PSNE in this game:

$$PSNE = \{(A, b), (B, a)\}$$

and the payoffs are $(4, 4)$ and $(1, 2)$ respectively.

For the modified question, it is then enough to observe that there is a mixed NE where P1 uses A and B and P2 uses a and b. We use a guess and verify approach: guess that P1 plays A w.p. p and B w.p. $1 - p$ and that P2 plays a w.p. q and b w.p. $1 - q$. Then, we solve for (p, q) that makes the players indifferent and after that argue that we have found a NE by showing that players do not want to deviate to C or c.

P1 is indifferent between A and B if

$$4(1-q) = q \Rightarrow q = \tfrac{4}{5}.$$

P2 is indifferent between a and b if

$$2(1-p) = 4p \Rightarrow p = \tfrac{1}{3}.$$

Now $(p = 1/3, q = 4/5)$ is a NE if neither player wants to deviate to C/c. If players follow $(p = 1/3, q = 4/5)$, they get $u_1 = 4/5$ and $u_2 = 4/3$. Bu deviating to C, P1 gets $2/5$ and by deviating to c, P2 gets $4/3$. Neither has a strictly profitable deviation and hence we have found a third NE, with payoffs $(4/5, 4/3)$.

Everything that follows is EXTRA:

Before we find all mixed strategy NE, we see that it is possible to construct a mixed strategy that would yield a strictly higher expected payoff than C. Assume that P1 plays A with probability p, B with probability z and C with probability $1-p-z$. Now consider the e.g. strategy defined as $\sigma_1 : (p, z) = (\frac{4}{5}, \frac{1}{5})$. σ_1 yields the expected payoff vector $(\frac{1}{5}, \frac{16}{5}, \frac{26}{5})$ which is strictly higher than what C yields, thus we can use IESDS to eliminate C. No strategies can be eliminated by IESDS for P2.

Now assume P1 plays A and B with probability p and $1-p$ respectively, and that P2 plays a, b and c with probability q, r and $1-q-r$ respectively. P1 is indifferent between A and B if

$$4r + 6(1-q-r) = q + 2(1-q-r) \Rightarrow q = \tfrac{4}{5}$$

For P2, the expected utility from playing a, b and c is

$$E\left[u_2(a))\right] = 0p + 2(1-p) = 2 - 2p$$

$$E\left[u_2(b))\right] = 4p + 0(1-p) = 4p$$

$$E\left[u_1(c))\right] = 0p + 2(1-p) = 2 - 2p$$

P2 is always indifferent between playing a and c and is indifferent between b and either a or c if

$$2-2p=4p \Rightarrow p=\tfrac{1}{3}$$

We can now write up the best response functions for the two players:

$$BR_1(q,r)=\begin{cases} p=0 & \text{if } q>\frac{4}{5} \\ p\in[0,1] & \text{if } q=\frac{4}{5} \\ p=1 & \text{if } q<\frac{4}{5} \end{cases}$$

$$BR_2(p)=\begin{cases} (q,r)=(0,1) & \text{if } p>\frac{1}{3} \\ (q,r)=(q,r), q\in[0,1], r\in[0,1-q] & \text{if } p=\frac{1}{3} \\ (q,r)=(q,0), q\in[0,1] & \text{if } p<\frac{1}{3} \end{cases}$$

We then see that we have the following MSNE:

$$MSNE:(p^*,q^*,r^*)=$$
$$\left\{(1,0,1),\left[(\tfrac{1}{3},\tfrac{4}{5},r), r\in\left[0,\tfrac{1}{5}\right]\right],\left[(0,q,0), q\in\left[\tfrac{4}{5},1\right]\right],\left[(p,\tfrac{4}{5},0), p\in[0,\tfrac{1}{3}]\right\}^*\right.$$

(*including the PSNE)

(b) Suppose that the above described stage-game is played twice (the players observe each other's action choices for the first round before the second round). A player's payoff is the undiscounted sum of the payoffs form both rounds. Is this a game of perfect information? (No need to draw the game tree).

Solution: No. A game has perfect information if each player, when making any decision, is perfectly informed of all the events that have previously occurred. The players in this game only observe what happened in the previous [first] stage after they got to make a choice [being unaware of what the other did at the time].

(c) Consider the twice-played game. Construct some SPE strategies such that (C,c) is played in the first stage. What are the expected payoffs in that equlibrium? Compare with part (a). (Hint. Use a bad stage NE as the 'punishment' and a good NE as the 'reward'.)

Solution: We have a finitely repeated game whose stage game has 2 PSNE with different payoffs and numerous MSNE. (C, c) is not a NE so we have to construct a strategy that will incentivize players to cooperate in the first stage. In order to do that, we need a reward (in case they manage to coordinate) and a punishment (in case any deviation takes place). Using a low-payoff NE is a punishment is also known as a "Nash Threat". Given that in the second (and last) stage a NE has to be played both on (if they coordinate - the reward) and off (if someone deviates - the punishment) the equilibrium path(EP), we can use the high-payoff NE (A, b) as a reward and the low-payoff NE (B, a) as the punishment/threat (any MSNE could also be used as a threat). We construct the following trigger strategies:

Trigger strategy for P1 (P2):
In stage 1, play C (c)
In stage 2, play A (b) if the outcome of stage 1 was (C, c), otherwise play B (a)

For these TS to constitute a SPNE, we need them to constitute a NE in all subgames both on and off the EP. To check if we have a NE on the EP, we can construct the following reduced bi-matrix

		P2		
		a	b	c
	A	0+1, 0+2	4+1, 4+2	6+1, 0+2
P1	B	1+1, 2+2	0+1, 0+2	2+1, 2+2
	C	0+1, 6+2	2+1, 2+2	5+4, 5+4

By plotting in best responses, we clearly see, that following the TS will constitute a NE on the EP. Since all deviations will result in the players playing (B, a), in the second stage, we can conclude that the TS also constitutes NE in all subgames off the EP. Thus, we can conclude that the TS constitutes a SPNE and that the expected payoff is $(9, 9)$.

(d) EXTRA. Consider again the twice played game but now with discounting. Show that you can support (C, c) in the first round with a lower discount factor if you use the mixed strategy NE as the punishment.

Solution: If we include a discount factor and follow the TS defined in problem 2(c), P1 will have no incentive to deviate if:

$$5 + 4\delta \geq 6 + 1\delta \geq\Rightarrow \delta \geq \tfrac{1}{3}$$

P2 will have no incentive to deviate if:

$$5 + 4\delta \geq 6 + 2\delta \geq\Rightarrow \delta \geq \tfrac{1}{2}$$

Thus, no player will have incentive to deviate if $\delta \geq \frac{1}{2}$

If instead of (B, a) we used the MSNE where P1 mixes between A (w.p. 1/3) and B and P2 mixes between a (w.p. 4/5) and b. P1 would have no incentive to deviate in the first period if

$$5 + 4\delta \geq 6 + \tfrac{4}{5}\delta \geq\Rightarrow \delta \geq \tfrac{5}{16}$$

P2 will have no incentive to deviate if:

$$5 + 4\delta \geq 6 + \tfrac{4}{3}\delta \geq\Rightarrow \delta \geq \tfrac{3}{8}$$

Thus, no player will have incentive to deviate if $\delta \geq \frac{3}{8}$ in the first period.

Clearly, using the above mentioned MSNE as the threat instead of the PSNE, (C, c) can be supported with a lower discount factor. (No one wants to deviate in either SPE in the second period because they are playing a NE after which the game ends.)

3. Consider again the same stage game as in Question 2. Now, assume that the game is played infinitely many times.

(a) Construct a trigger strategy profile where (C, c) is always played on the equilibrium path. (There is not only one right answer: you can choose the punishment many different ways.)

Solution: We now have an infinitely repeated game, which entails, that there is no last stage, i.e. we no longer have to play a stage game NE (SGNE) in the last stage. However, using a SGNE as the threat in a trigger strategy ensures that the players always plays a NE in all subgames off the EP, so the SGNE are still very useful. Thus, we can construct trigger strategies for the players that are very similar to the trigger strategies constructed in 2(c). An example (of many) could be:

Trigger strategy for P1 (P2):
In stage 1, play C (c)
In stage 2+, play C (c) if the outcome of all previous stages was (C, c), otherwise play B (a)

(b) For the trigger strategy profile in part (a), find the smallest discount factor such that the profile gives a SPE. (Remember to argue both on path and off path.)

Solution: A player will not deviate from the TS if the sum of the discounted payoff from following the TS is at least as good the the sum of discounted payoff from deviating. If P1 (P2) follows the TS, she will receive a payoff of 6 in each stage. If P1 was to deviate from the TS, the only profitable deviation would be to play A (a) and receive a payoff of $6 > 5$. The deviation will however result in (B, a) being played in all subsequent stages, with a payoff of 1 (2). Thus, P1 will not deviate from the TS if:

$$\sum_{t=1}^{\infty} 5\delta^{t-1} \geq 6 + \sum_{t=2}^{\infty} 1\delta^{t-1} \Rightarrow \frac{5}{1-\delta} \geq 6 + \frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{1}{5}$$

While P2 will not deviate from TS if

$$\sum_{t=1}^{\infty} 5\delta^{t-1} \geq 6 + \sum_{t=2}^{\infty} 2\delta^{t-1} \Rightarrow \frac{5}{1-\delta} \geq 6 + \frac{2\delta}{1-\delta} \Rightarrow \delta \geq \frac{1}{4}$$

Thus, as long as $\delta \geq \frac{1}{4}$, we have a NE on the EP. The players move off the EP, if one of the players deviates from playing (C, c). The TS then prescribes playing (B, a) in all subsequent stages, and since (B, a) is a SGNE, we have ensured that the players are also playing a NE in all subgames off the EP. Thus, we can conclude that the above mentioned TS will constitute a SPNE when $\delta \geq \frac{1}{4}$