Microeconomics III

Fall 2021

**Assignment 3: Solutions**

1. **Auctions.** Consider the following common value auction. There are two bidders, 1 and 2, whose types $\theta_i$ for both $i \in 1, 2$, are independently drawn from a uniform distribution on $[0, 100]$. The value of the object to both bidders is the sum of the types, i.e. $\theta_i + \theta_j$. The object is offered for sale in a first price auction. Hence the payoffs depend on the bids $b_i$ and types as follows (assume a coin toss if $b_i = b_j$):

$$u_i(b_i, b_j, \theta_i, \theta_j) = \begin{cases} \theta_i + \theta_j - b_i \text{ if } b_i > b_j, \\ \frac{1}{2}(\theta_i + \theta_j - b_i) \text{ if } b_i = b_j, \\ 0 \text{ otherwise.} \end{cases}$$

   (a) Show that strategies $s_i(\theta_i) = \theta_i$ for $i = 1, 2$ (i.e. the players bid their own type) form a Bayes Nash equilibrium in this game.

   **Solution:** Let's calculate the best-response of player $i$ given that $j \neq i$ plays the linear strategy $b_j = \theta_j$. The expected utility of player $i$ equals

$$\begin{aligned} Eu_i(b_i, b_j, \theta_i, \theta_j) &= \Pr(b_i > b_j(\theta_j))(\theta_i + E(\theta_j | b_i > b_j(\theta_j)) - b_i) \\ &= \Pr(b_i > \theta_j)(\theta_i + E(\theta_j | b_i > \theta_j) - b_i) \\ &= \left(\frac{b_i - 0}{100 - 0}\right)(\theta_i + \frac{0 + b_i}{2} - b_i) \\ &= \frac{b_i}{100}(\theta_i + \frac{b_i}{2} - b_i) \end{aligned}$$

   Taking the first order condition with regards to $b_i$ yields

$$\frac{1}{100}(\theta_i + \frac{b_i}{2} - b_i) + \frac{b_i}{100}(\frac{1}{2} - 1) = 0 \Leftrightarrow b_i = \theta_i.$$

   That is, it is a best response to play $b_i = \theta_i$ against $b_j = \theta_j$.
   Thus, there exist a symmetric BNE in linear strategies, where $s_i^*(\theta_i) = b_i^*(\theta_i) = \theta_i$

   (b) If $\theta_i = 1$, the equilibrium bid is 1, but it might seem that the expected value of the object is $1 + 50 = 51$. Why doesn't the bidder behave more aggressively?

   **Solution:** The players only get the object if their bid is winning. They should consider only the expected value of the object conditional on their bid being the winning bid. In the equilibrium, the expected value of the object conditional on winning equals $\theta_i + E(\theta_j \mid b_i > b_j) = \theta_i + \frac{b_i}{2}$. So, if $\theta_i = 1$, the expected value conditional on winning is 1.5 although the unconditional expected value is 51.

   (c) Assume now that values are private: the value of the object to bidder $i$ is $2\theta_i$ for $i = 1, 2$. Solve the BNE where players use linear strategies (Hint. we covered FPA with private values in the lectures and in PS9).

   **Solution:** Now the values are private, and player $i$ values the object $v_i = 2\theta_i$, i.e $v_i \sim U(0, 200)$. Let's calculate the best-response of player $i$ given that $j \neq i$ plays the linear strategy $b_j = cv_j$. The expected utility of player $i$ equals

$$\begin{aligned} Eu_i(b_i, b_j, v_i) &= \Pr(b_i > b_j(v_j))(v_i - b_i) \\ &= \Pr(b_i > cv_j)(v_i - b_i) \\ &= \Pr(\frac{b_i}{c} > v_j)(v_i - b_i) \\ &= (\frac{\frac{b_i}{c} - 0}{200 - 0})(v_i - b_i) \\ &= \frac{b_i}{200c}(v_i - b_i) \end{aligned}$$

   Taking the first order condition with regards to $b_i$ yields

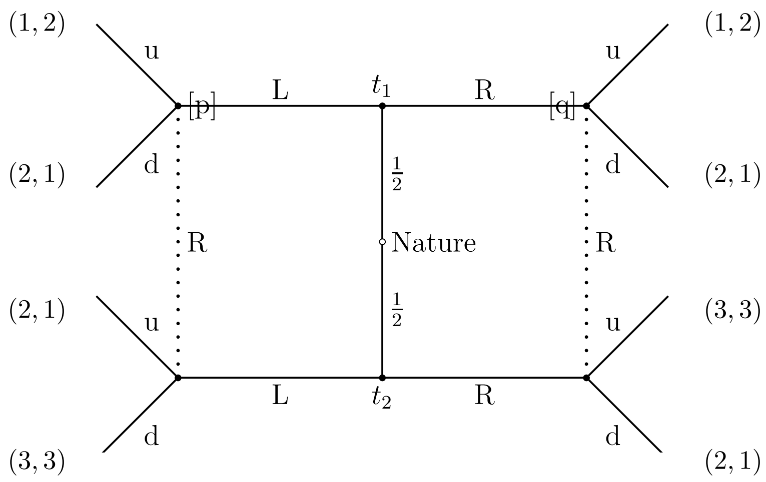
$$\frac{1}{200c}(v_i - b_i) + \frac{b_i}{200c}(-1) = 0 \Leftrightarrow b_i = \frac{1}{2}v_i.$$

Clearly $c = \frac{1}{2}$, so it is a best response to play $b_i = \frac{1}{2}v_i$ against $b_j = \frac{1}{2}v_j$.
Thus, there exist a symmetric BNE in linear strategies, where $b_i^*(v_i) = \frac{1}{2}v_i$

(d) Compare the equilibria in (a) and in (c). Are the bidders better off when values are common or when they are private? What is the intuition? (informal discussion is enough)

**Solution:** With private and common values, the bidders will bid the same, since $b_i^*(v_i) = \frac{1}{2}v_i = \frac{1}{2}2\theta_i = \theta_i = b_i^*(\theta_i)$. However, the bidders are worse off with common values, since the expected value conditional on winning is lower with common values. E.g. if player $i$ draws $\theta_i = 80$, bids 80 **and** wins, with common values, the expected value of the object is $\theta_i + E(\theta_j \mid b_i > b_j) = \theta_i + \frac{b_i}{2} = 80 + \frac{80}{2} = 120$. On the other hand, with private values, if player $i$ draws $\theta_i = 80$, they will value the object $2\theta_i = 2 * 80 = 160$. Competition between the two bidders is more severe under common values.

2. **Signaling.** Solve for all pure strategy PBE for the following signaling game by following the cookbook from Lecture 10.




**Solution.** We check each possible strategy one after one to see which constitutes a Perfect Bayesian equilibrium.

**Type $t_1$ plays L, and type $t_2$ plays R:**
We find, due to Bayesian updating, that $p = 1$ and $q = 0$. The optimal actions of the receiver can now be derived. We have

$$u_R(L, u; p = 1) = 2$$
$$u_R(L, d; p = 1) = 1$$
$$u_R(R, u; q = 0) = 3$$
$$u_R(R, d; q = 0) = 1.$$

We see the receiver will play u if she receives the message L and u if seeing the message R.

$$a(L) = u$$
$$a(R) = u.$$

For this to be a PBE the sender must not want to deviate.

$$u_S(L, u; t_1) = 1 = u_S(R, u; t_1) = 1$$
$$u_S(R, u; t_2) = 3 > u_S(L, u; t_2) = 2.$$

As no type wants to deviate we have a PBE. PBE= $\{LR, uu; p = 1, q = 0\}$.

**Type $t_1$ plays R, and type $t_2$ plays L:**
We find, due to Bayesian updating, that $p = 0$ and $q = 1$. The optimal actions of the receiver can now be derived. We have

$$u_R(L, u; p = 0) = 1$$
$$u_R(L, d; p = 0) = 3$$
$$u_R(R, u; q = 1) = 2$$
$$u_R(R, d; q = 1) = 1.$$

We see the receiver will play u if she receives the message L and u if seeing the message R.

$$a(L) = d$$
$$a(R) = u.$$

We see the sender has an interest in deviating if he is a type $t_1$, as he can do better by sending the message L instead of R

$$u_S(R, u; t_1) = 1 < u_S(L, d; t_1) = 2.$$

This strategy is therefore not a PBE.

**Both types play L:**
We find, due to Bayesian updating, that $p = \frac{1}{2}$ . The optimal actions of the receiver can now be derived. We have

$$u_R(L, u; p = \frac{1}{2}) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}$$
$$u_R(L, d; p = \frac{1}{2}) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$$

We see the receiver will play d if she receives the message L.

$$a(L) = d.$$

For this to be a PBE the sender must not want to deviate. Neither type can potentially do better by deviating so this is an equilibrium. We, however, still need to specify the action of the receiver given her beliefs if she receives the message R. We see the receiver will always play u if she receives the message R.

$$u_R(R, u; q) = q2 + (1 - q)3 = 3 - q$$
$$u_R(R, d; q) = q + (1 - q) = 1.$$
$$u_R(R, d; q) < u_R(R, u; q)$$
$$a(R) = u$$

As no type wants to deviate we have a PBE. PBE= $\{LL, du; p = \frac{1}{2}, q \in [0, 1]\}$.

**Both types play R:**
We find, due to Bayesian updating, that $q = \frac{1}{2}$ . The optimal actions of the receiver can now be derived. We have

$$u_R(R, u; q = \frac{1}{2}) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 3 = \frac{5}{2}$$
$$u_R(R, d; q = \frac{1}{2}) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$$

We see the receiver will play u if she receives the message R.

$$a(R) = u.$$

For this to be a PBE the sender must not want to deviate. Type $t_1$ can potentially do better by deviating and playing L if the receiver chooses u. We must specify the conditions such that this is not the case.

$$\begin{aligned} u_R(L, u; p) &= p2 + (1-p)1 = 1 + p \\ u_R(L, d; p) &= p + (1-p)3 = 3 - 2p. \\ u_R(L, u; p) &\geq u_R(L, d; p) \\ 1 + p \geq 3 - 2p &\Leftrightarrow p \geq \frac{2}{3}. \end{aligned}$$

For these beliefs the receiver will play u if she sees L.

$$a(L) = u.$$

As no type wants to deviate we have a PBE. $PBE = \{RR, uu; p \in [\frac{2}{3}, 1], q = \frac{1}{2}\}$.
You could also write up all the pure strategy PBE in one set.

$$PBE = \left\{ \left( LR, uu; p = 1, q = 0 \right); \left( LL, du; p = \frac{1}{2}, q \in [0, 1] \right); \left( RR, uu; p \in [\frac{2}{3}, 1], q = \frac{1}{2} \right) \right\}.$$