Macroeconomics III - Lecture 2

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September 16, 2021

Multiple periods

- Problem

$$\max_{c_0,\dots,c_T,a_1,\dots,a_{T+1}} \quad \sum_{t=0}^{T} \beta^t u(c_t),$$
$$\text{s.t. } a_{t+1} = a_t R_t + w_t - c_t, \;\; a_0 R_0 \text{ given, } a_{T+1} \geq 0$$

- Immediate option (conjecturing that $a_{T+1} = 0$):

$$\max_{a_1,\dots,a_T} \sum_{t=0}^{T} \beta^t u(a_t R_t + w_t - a_{t+1}),$$
$$\text{s.t. } a_0 R_0 \text{ given, } a_{T+1} = 0$$

Multiple periods

Using the IBC

- Approach the problem by constructing the Lagrangian through the IBC
- Denote prices as of time 0 by $q_t \equiv (R_1 R_2 \dots R_t)^{-1}$, with $q_0 \equiv 1$
- Take the IBC at the final period:

$$
\begin{aligned}
a_{T+1} &= a_T R_T + w_T - c_T \\
&= (a_{T-1} R_{T-1} + w_{T-1} - c_{T-1}) R_T + w_T - c_T \\
&= \dots \\
&= a_0 R_0 R_1 \dots R_T + (w_0 - c_0) R_1 R_2 \dots R_T + (w_1 - c_1) R_2 R_3 \dots R_T + \\
&\quad + \dots + (w_{T-1} - c_{T-1}) R_T + w_T - c_T
\end{aligned}
$$

Multiple periods

Using the IBC

- Multiplying the IBC by q_T and exploiting $a_{T+1} = 0$:

$$
\begin{aligned}
q_T a_{T+1} &= a_0 R_0 R_1 ... R_T q_T + \\
&+ q_T \left[(w_0 - c_0) R_1 ... R_T + (w_1 - c_1) R_2 ... R_T + ... + (w_{T-1} - c_{T-1}) R_T + w_T - c_T \right]
\end{aligned}
$$

$$
\begin{aligned}
q_T a_{T+1} &= a_0 \underbrace{\frac{R_0 R_1 ... R_T}{R_1 R_2 \dots R_T}}_{=R_0} + (w_0 - c_0) \underbrace{\frac{R_1 R_2 ... R_T}{R_1 R_2 \dots R_T}}_{=1} + \\
&+ (w_1 - c_1) \underbrace{\frac{R_2 R_3 ... R_T}{R_1 R_2 \dots R_T}}_{=R_1^{-1} = q_1} + ... + \\
&+ (w_{T-1} - c_{T-1}) \underbrace{\frac{R_T}{R_1 R_2 \dots R_T}}_{=(R_1 R_2 ... R_{T-1})^{-1} = q_{T-1}} + (w_T - c_T) \underbrace{\frac{1}{R_1 R_2 \dots R_T}}_{=q_T}
\end{aligned}
$$

Multiple periods

Using the IBC

- Thus

$$q_T a_{T+1} = 0 = a_0 R_0 + \sum_{t=0}^{T} q_t (w_t - c_t)$$

- ...and

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda [a_0 R_0 + \sum_{t=0}^{T} q_t (w_t - c_t)]$$

Multiple periods

Using the IBC

- Differentiating w.r.t. $c_0, c_1, ..., c_T$ yields the FOCs

$$\beta^t u'(c_t) = \lambda q_t, \ t = 0, \ldots, T.$$

- Combining the t and $t+1$ FOCs yields the Euler equation

$$u'(c_t) = \beta R_{t+1} u'(c_{t+1})$$

- As in the two-period case, the Euler equation characterizes the slope of the optimal consumption path
- To find consumption levels we must combine the Euler equations with the IBC

Multiple periods

Dynamic budget constraints + terminal condition

- In alternative, we can use the original dynamic budget constraint (DBC) and the terminal condition:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda_t [a_{t+1} - a_t R_t - w_t + c_t] + \mu a_{T+1}$$

- Thus:

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{T} \beta^t u(c_t) + \lambda_t [a_{t+1} - a_t R_t - w_t + c_t] + \mu a_{T+1} \\
&= \beta^0 u(c_0) + \beta^1 u(c_1) + ... + \beta^T u(c_T) + \\
&\quad + \lambda_0 [a_1 - a_0 R_0 - w_0 + c_0] + \lambda_1 [a_2 - a_1 R_1 - w_1 + c_1] + \\
&\quad + ... + \lambda_T [a_{T+1} - a_T R_T - w_T + c_T] + \\
&\quad + \mu a_{T+1}
\end{aligned}$$

Multiple periods

Dynamic budget constraints + terminal condition

- The FOCs w.r.t. $c_0, c_1, ..., c_T$ and $a_1, ..., a_T$ are:

$$
\begin{aligned}
\beta^t u'(c_t) &= \lambda_t, \; t = 0, \ldots, T \\
\lambda_t &= \lambda_{t+1} R_{t+1}, \; t = 0, \ldots, T-1
\end{aligned}
$$

- The FOC w.r.t. a_{T+1} is $\lambda_T = \mu$, and the complementary slackness condition is $\mu a_{T+1} = 0$
- Combining the FOCs for consumption, once again, yields:

$$
u'(c_t) = \beta R_{t+1} u'(c_{t+1})
$$

- Moreover, non-satiation (i.e., $u' > 0$) implies that $\lambda_T = \mu > 0$, so that $a_{T+1} = 0$ (i.e., the transversality condition we argued before)

Infinite horizon

- There are three reasons to consider the limiting case $T \to \infty$:
    - Intergenerational altruism
    - Time-invariant survival probability
    - Mathematical simplicity

Infinite horizon

No Ponzi game condition

- To derive the IBC we need to specify the terminal condition. Note that $\lim_{T\to\infty} a_{T+1} \geq 0$ would be unnecessarily tight
- Instead the appropriate constraint is the "*no Ponzi game condition*" (NPGC):

$$\lim_{T\to\infty} q_T a_{T+1} \geq 0$$

- This allows holding debt in the long run, but prevents household from permanently rolling it over and never servicing it

Infinite horizon

- Using the NPGC we can derive the IBC:

$$a_0 R_0 + \lim_{T \to \infty} \sum_{t=0}^{T} q_t (w_t - c_t) = \lim_{T \to \infty} q_T a_{T+1} \geq 0.$$

- Proceeding as before (i.e., setting $\lim_{T \to \infty} q_T a_{T+1}$ as small as possible), allows us to write the Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda [a_0 R_0 + \sum_{t=0}^{\infty} q_t (w_t - c_t)]$$

- Differentiating we get the same FOCs as before

$$\beta^t u'(c_t) = \lambda q_t, \ \ t = 0, 1, \ldots,$$

and, thus, the Euler equation...

Ramsey model

- We can now embed our microfounded model of consumption-saving behavior in a general equilibrium model of capital accumulation
- For this we add a firm sector and impose market clearing
- The first framework we are going to detail is known as the Ramsey model
- We assume the economy is populated by a continuum of identical households of mass one
- This *representative agent* has an infinite planning horizon

Ramsey model

Aggregation

- The representative-agent assumption makes the aggregation of individual choices trivial
- Since households are all alike and the economy is closed, the assets they accumulate correspond to the physical capital stock in the economy ($S = I$):

$$k_t = a_t$$

- Capital depreciates at rate δ per period. Thus the return R_t on household savings equals the rental rate on capital paid by firms, r_t, plus the undepreciated capital:

$$R_t = r_t + 1 - \delta$$

Ramsey model

Firms

- Firms compete and take rental rates and wages as given. The representative firm maximizes profits:

$$\max_{K_t, L_t} f(K_t, L_t) - r_t K_t - w_t L_t$$

- FOCs

$$\begin{aligned} f_K(K_t, L_t) &= r_t \\ f_L(K_t, L_t) &= w_t \end{aligned}$$

which define the demand functions for capital and labor

Ramsey model

Dynamics

- Capital accumulation is determined from the dynamic budget constraint and the fact that optimal consumption satisfies the Euler equation

$$\begin{aligned} a_{t+1} &= a_t(1 + r_t - \delta) + w_t - c_t + z_t \\ u'(c_t) &= \beta(1 + r_{t+1} - \delta)u'(c_{t+1}) \end{aligned} \tag{1}$$

- ...and the *transversality condition* (TVC)

$$\lim_{T\to\infty} q_T k_{T+1} = 0 \Leftrightarrow \lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$$

Ramsey model

Market clearing

- Market clearing implies

$$\begin{aligned} L_t &= 1 \\ K_t &= k_t = a_t \end{aligned}$$

- This implies the following resource constraint

$$\begin{aligned} a_{t+1} &= a_t(1 + r_t - \delta) + w_t - c_t + \underbrace{f(K_t, L_t) - r_t K_t - w_t L_t}_{=z_t} \\ k_{t+1} &= k_t(1 + r_t - \delta) + w_t - c_t + f(k_t, 1) - r_t k_t - w_t \\ k_{t+1} &= k_t(1 - \delta) + f(k_t, 1) - c_t \qquad (2) \end{aligned}$$

- Intuition:

$$f(k_t, 1) = k_{t+1} - k_t(1 - \delta) + c_t$$

Ramsey model

Laws of motion

- Laws of motion for capital and consumption:

$$
\begin{aligned}
k_{t+1} &= k_t(1-\delta) + f(k_t, 1) - c_t \\
u'(c_t) &= \beta(1 + f_K(k_{t+1}, 1) - \delta)u'(c_{t+1})
\end{aligned}
$$

- Note that, given k_0, these equations pin down k_{t+1} and c_{t+1}, conditional on the initial value of consumption, c_0 (will get back to this later on)

Ramsey model

Analysis

- Now we will perform a graphical analysis of the economy's dynamics
- To do this we plot in a k, c phase diagram the curves (*loci*) that correspond to $c_{t+1} = c_t = c$ and $k_{t+1} = k_t = k$, i.e. the combinations of k and c that respectively imply no time change for these variables:

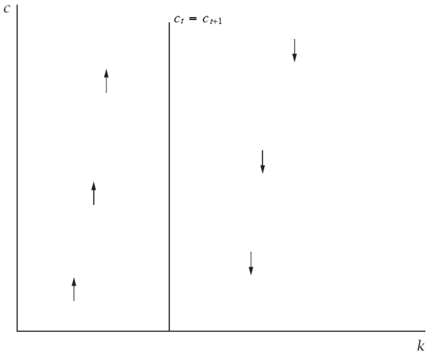
$$
\begin{aligned}
c &= f(k,1) - \delta k \\
1 &= \beta(1 + f_K(k,1) - \delta)
\end{aligned}
$$

- Their intersection defines the steady state, k^*, c^*. How do c and k move outside these curves? For any initial allocation, is the steady state always attained?

Ramsey model

Analysis

Take first the locus for $c_t = c_{t+1} = c$:

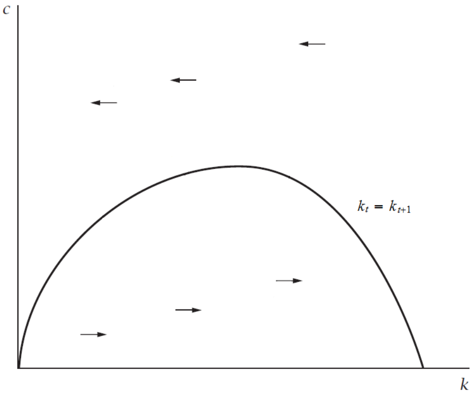


- High (low) level of capital⇒low (high) marginal product⇒low (high) rate of interest
- If the interest rate is relatively low, we'd rather bring consumption forward, so future consumption growth falls
- By contrast, if the interest rate is relatively high, we'd rather postpone consumption, so future consumption growth rises

Ramsey model

Analysis

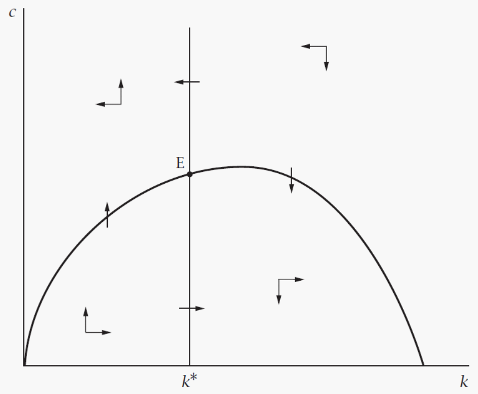
- Take the locus for $k_t = k_{t+1} = k$:



- High consumption⇒little output left to invest⇒capital falls
- Due to concavity, at high enough k_t, $k_{t+1} < k_t$ even with low c_t

Ramsey model

Analysis



- We can combine both loci for the complete phase diagram
- There is a *balanced growth path (BGP)* at point **E**: c and k are constant at their steady-state level, c^* and k^*
- Arrows suggest that we may converge to BGP if we start somewhere in NE or SW quadrant

Ramsey model

Analysis

- According to Solow growth model, $f_K(k^{gr}, 1) = \delta$ defines the level of capital that maximizes per-capita consumption along the BGP: $\max_k c = f(k) - \delta k$. Same situation here
- Note that k^* must be below the *Golden Rule* level (why?)

$$f_K(k^*, 1) = \delta + \frac{1}{\beta} - 1 > \delta = f_K(k^{gr}, 1)$$

- Define **E** as the *Modified Golden Rule* equilibrium: $k^* = k^{mgr}$

Ramsey model

Balanced growth path

- Once the economy reaches the steady state, same dynamics as in the Solow growth model
- The only difference is that steady-state capital (k^{mgr}) is lower than the gr-level (k^{gr})
- The reason for this is that saving is the result of optimizing behavior by households that value consumption-utility intertemporally and, absent externalities, they would never choose a level of capital above the golden rule level
- This is not the case in Solow, where MPC/MPS is exogenous

Ramsey model

Welfare

- A natural question is whether the equilibrium of this economy represents a desirable outcome
- F*irst welfare theorem:* if markets are competitive and complete and there are no externalities (and if the number of agents is finite), then the decentralized equilibrium is Pareto-efficient—that is, it is impossible to make anyone better off without making someone else worse off
- Since the conditions of the first welfare theorem hold in the Ramsey model, the equilibrium must be Pareto-efficient

Ramsey model

Welfare

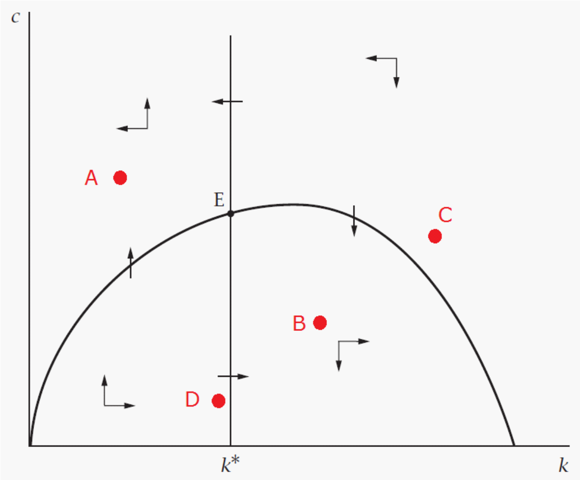
- To see this, consider the problem facing a social planner who can dictate the division of output between consumption and investment at each date and who wants to maximize the lifetime utility of a representative household
- This problem is identical to that of an individual household except that the paths of w_t and r_t are not taken as given (prove it):

$$\max_{c_t, k_{t+1}} \quad \sum_{t=0}^{T} \beta^t u(c_t),$$

$$\text{s.t. } f(k_t, 1) = k_{t+1} - k_t(1-\delta) + c_t, \; k_0 \text{ given}, \; k_{t+1} \geq 0$$

Ramsey model

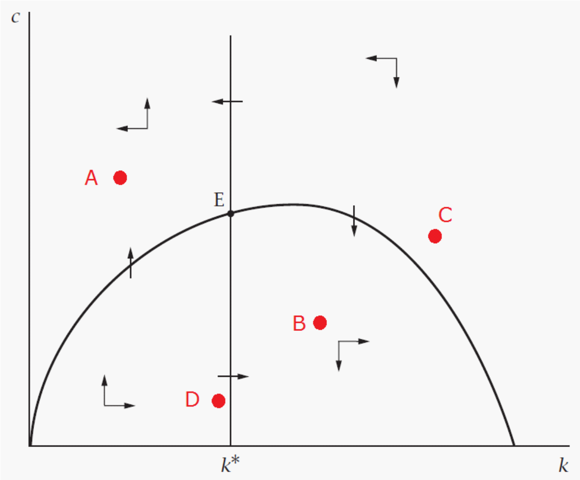
Analysis



- Suppose we start at point **A**

Ramsey model

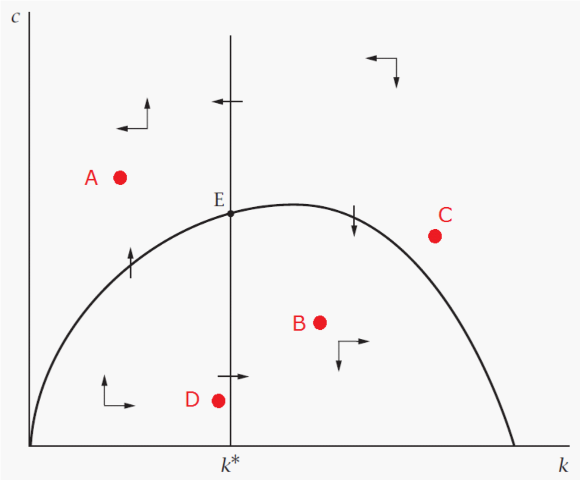
Analysis



- Suppose we start at point **B**

Ramsey model

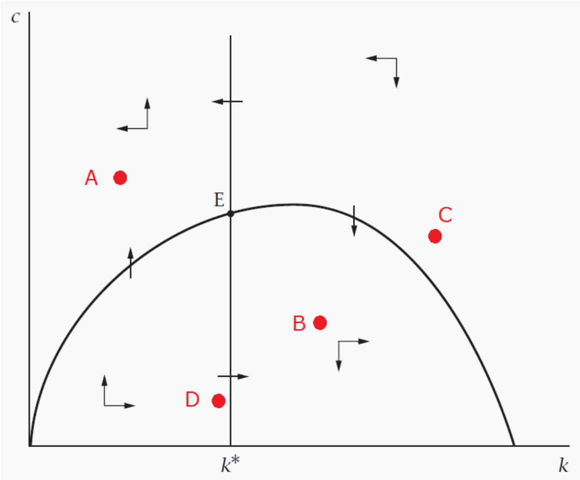
Analysis



- Suppose we start at point **C**

Ramsey model

Analysis

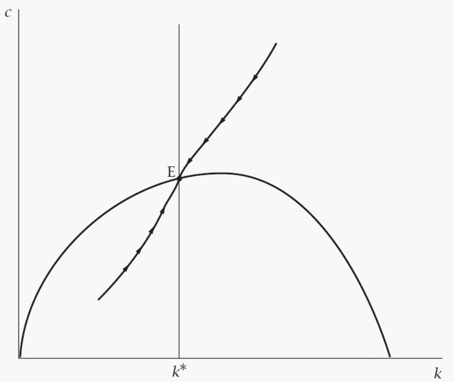


- Suppose we start at point **D**

Ramsey model

Analysis

For a given initial k_0 we can rule out all initial c_0 except one. Repeating this logic for all k_0 gives the saddle path



- For any k_0 there exists a unique saddle path such that the economy converges
- EE, LOM for capital and TVC hold at every point along this path
- Will not prove existence, but uniqueness follows once we pin down c_0

Ramsey model

Analysis

- Assume CRRA preferences, and let's compute c_0. Iterating the Euler equation we get

$$c_t = \left(\frac{\beta^t}{q_t}\right)^{\frac{1}{\sigma}} c_0$$

- Substituting in the IBC:

$$c_0 \sum_{t=0}^{\infty} \beta^{\frac{t}{\sigma}} q_t^{1-\frac{1}{\sigma}} = a_0 R_0 + \sum_{t=0}^{T} q_t w_t$$

Ramsey model

Effect of a rise in the discount factor

- Since β governs consumption preferences, changes in this parameter will affect the Euler equation
- The savings rate in Ramsey is endogenous, and determined by household trade-off between current and future consumption
- One parameter that directly affects how much we save is the discount rate (β^{-1}): If we care about the future more, everything else equal, we want to save more and consume less today
- As an exercise consider in a phase diagram the effect of a rise in β, assuming the economy is initially in the steady state

Ramsey model

Analysis