Macroeconomics III

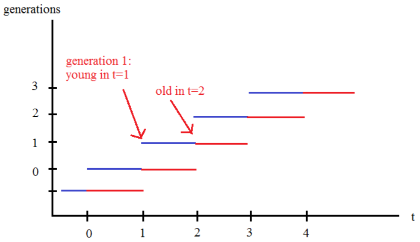
Lecture 4

Emiliano Santoro

University of Copenhagen

Overlapping generations model

- Second basic workhorse model in dynamic macroeconomics
- New ingredient: *(exogenous) population turnover*



- How does this affect aggregate savings? Welfare?
- Is there a role for the government to provide for the old (Social Security)?
- Does fiscal policy work differently in a setting with finite lives? How about government borrowing?

Overlapping generations model

- Due to and named after Peter Diamond, dates to 1965

- Builds on ealier work by Samuelson - one of the textbook exercises goes through the original Samuelson model if you're interested
- Yet another Nobel prize winner (2010) but for work on unemployment, together with Mortensen and Pissarides

Outline

1. OLG model setup (DR 2.8)
2. Characterizing the solution (DR 2.9-2.10)
3. Dynamics in a well-behaved special case (DR 2.10)
4. Welfare (DR 2.11)

Outline

OLG model setup

- Time is discrete and infinite $t = 0, 1, 2, ...$
- The economy is populated by agents that live for 2 periods - enough to derive qualitative implications, generalizable but then we need a computer to solve
- L_t individuals are born in period t. Assume the population grows at constant rate n, so

$$L_t = L_{t-1}(1 + n)$$

- Note that this means that there are L_t young people and $L_{t-1} = L_t/(1 + n)$ old people alive in period t

Preferences

- Agents derive utility from consumption while alive:

$$U_t = u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1})$$

- c_{1t}: Consumption when young at time t
- c_{2t+1}: Consumption when old at time $t+1$
- ρ: Discount rate

Budget constraints

- Agents work when young. They supply 1 unit of labor inelastically at wage rate w_t
- They split the labor income between consumption c_{1t} and savings s_t
- They retire when old and just consume their gross savings
- Let r_{t+1} denote the interest rate between t and $t+1$
- Budget constraints in each period of life:

$$\begin{aligned} c_{1t} + s_t &= w_t \\ c_{2t+1} &= (1 + r_{t+1}) s_t \end{aligned}$$

- Lifetime budget constraint (substitute out s_t)

$$c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1}} = w_t$$

Maximization problem

- For every t, agents born at time t solve the following problem

$$\max_{c_{1t}, c_{2t+1}} u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1})$$

subject to

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t$$

- The initial old make no choices and fully consume their wealth

Firms and production

- Production is the same as before - CRS technology, competitive markets, profit-maximizing firms
- Abstract from capital depreciation. Solving for firms' problem gives us the solutions for factor prices (prove it for the Cobb-Douglas case):

$$\begin{aligned} r_t &= f'(k_t) \\ w_t &= f(k_t) - f'(k_t)k_t \end{aligned}$$

where $k \equiv \frac{K}{L}$ capital per worker (not per capita, since we now have non-working old)

Outline

Characterizing solution to household problem

- Set up the Lagrangian

$$\mathcal{L} = u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1}) + \lambda \left[w_t - c_{1t} - \frac{c_{2t+1}}{1+r_{t+1}} \right]$$

to find the FOCs

$$[c_{1t}:] \qquad 0 = u'(c_{1t}) - \lambda$$

$$[c_{2t+1}:] \qquad 0 = \frac{1}{1+\rho} u'(c_{2t+1}) - \lambda \frac{1}{1+r_{t+1}}$$

- Substitute out the Lagrange multiplier to get

$$u'(c_{1t}) = \frac{1+r_{t+1}}{1+\rho} u'(c_{2t+1})$$

The OLG-version of the Euler equation!

Euler equation

$$u'(c_{1t}) = \frac{1 + r_{t+1}}{1 + \rho} u'(c_{2t+1})$$

- Intuition? Describes how marginal utilities of consumption between youth and old age are optimally related
- We equate marginal cost of giving up a unit of consumption today with the marginal benefit of consuming it (plus interest) tomorrow

Euler equation

- Why equate?
- Suppose we don't

$$u'(c_{1t}) < \frac{1 + r_{t+1}}{1 + \rho} u'(c_{2t+1})$$

- Then we can reduce c_{1t} by a small amount Δ, losing utility $u'(c_{1t})\Delta$
- We invest it and consume the proceeds tomorrow, gaining $\frac{1+r_{t+1}}{1+\rho} u'(c_{2t+1})\Delta$
- We just made a net utility gain! So we could not have been optimizing when $u'(c_{1t}) < \frac{1+r_{t+1}}{1+\rho} u'(c_{2t+1})$
- By the same argument, it can't be that $u'(c_{1t}) > \frac{1+r_{t+1}}{1+\rho} u'(c_{2t+1})$
- Thus, $u'(c_{1t}) = \frac{1+r_{t+1}}{1+\rho} u'(c_{2t+1})$ at the optimum

Euler equation

- If we assume CRRA utility

$$u(c) = \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma} & ,\sigma \neq 1 \\ \log c & ,\sigma = 1 \end{cases}$$

we can derive a relationship for the *levels* of consumption:

$$\begin{aligned} u'(c_{1t}) &= \frac{1+r_{t+1}}{1+\rho} u'(c_{2t+1}) \\ c_{1t}^{-\sigma} &= \frac{1+r_{t+1}}{1+\rho} c_{2t+1}^{-\sigma} \\ \frac{c_{2t+1}}{c_{1t}} &= \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\sigma}} \end{aligned}$$

- Consumption grows over the lifecycle when interest rates are high relative to the discount rate
- This effect is stronger, the more willing agents are to substitute consumption over time (high IES, low σ)

From consumption to savings

- What can we say about optimal savings, both at the individual and the aggregate level?
- Combine the budget constraints

$$
\begin{aligned}
c_{1t} + s_t &= w_t \\
c_{2t+1} &= (1 + r_{t+1})s_t
\end{aligned}
$$

with the Euler equation

$$u'(c_{1t}) = \frac{1 + r_{t+1}}{1 + \rho} u'(c_{2t+1})$$

by substituting out consumption

$$u'(w_t - s_t) = \frac{1 + r_{t+1}}{1 + \rho} u'[(1 + r_{t+1})s_t]$$

- This implicitly defines optimal savings as a function of the wage and the interest rate, $s(w_t, r_{t+1})$

Characterizing optimal savings

$$u'(w_t - s_t) = \frac{1 + r_{t+1}}{1 + \rho} u'[(1 + r_{t+1})s_t]$$

- Effect of interest rate on savings is in general ambiguous
  - Substitution effect: $\frac{\partial s}{\partial r} > 0$. Save more because of higher return
  - Income effect: $\frac{\partial s}{\partial r} < 0$. Save less because you are richer

A note on aggregation

- Once we know individual decisions, what are the aggregates in the OLG setting?
- Aggregate savings? As only the young save, and there are L_t of them:

$$S_t = s_t L_t$$

- Aggregate capital stock in $t+1$: (i) saving by the young, (ii) dissaving by the old, (iii) un-depreciated capital carried over from t:

$$\begin{aligned} K_{t+1} &= S_t - (1-\delta)K_t + (1-\delta)K_t \\ K_{t+1} &= S_t \\ k_{t+1}(1+n) &= s_t \end{aligned}$$

Aggregate capital accumulation

- Aggregate capital accumulation is

$$k_{t+1}(1+n) \quad = \quad s(w_t, r_{t+1})$$

- We know equilibrium wage and interest rates as a function of capital per worker - substitute those in:

$$k_{t+1}(1+n) \quad = \quad s[f(k_t) - k_t f'(k_t), f'(k_{t+1})]$$

- This is an implicit law of motion for aggregate capital per worker: For given k_0, we can trace out the complete optimal path for $\{k_t\}_{t=1}^{\infty}$

Outline

A special case: Log CD

To make more progress solving for the equilibrium, we need to make assumptions on functional forms

Assumption 1 Logarithmic utility

$$u(c) = \log c$$

Assumption 2 Cobb-Douglas production function

$$F(K, L) = K^{\alpha} L^{1-\alpha}, \alpha \in (0, 1)$$

Log CD case: Consumption, savings and capital accumulation

- The Euler equation simplifies

$$
\begin{aligned}
u'(c_{1t}) &= \frac{1+r_{t+1}}{1+\rho} u'(c_{2t+1}) \\
\frac{1}{c_{1t}} &= \frac{1+r_{t+1}}{1+\rho} \frac{1}{c_{2t+1}} \\
c_{2t+1} &= \frac{1+r_{t+1}}{1+\rho} c_{1t}
\end{aligned}
$$

Log CD case: Consumption, savings and capital accumulation

- We can derive an *explicit* expression for optimal savings - start with previous expression:

$$
\begin{aligned}
u'(w_t - s_t) &= \frac{1 + r_{t+1}}{1 + \rho} u'((1 + r_{t+1})s_t) \\
\frac{1}{w_t - s_t} &= \frac{1 + r_{t+1}}{1 + \rho} \frac{1}{(1 + r_{t+1})s_t} \\
s_t &= \frac{1}{1 + \rho}(w_t - s_t) \\
s_t &= \frac{1}{2 + \rho} w_t
\end{aligned}
$$

  - Substitution and income effects of r on s cancel each other out with log utility
  - We save a *constant* fraction of labor income!

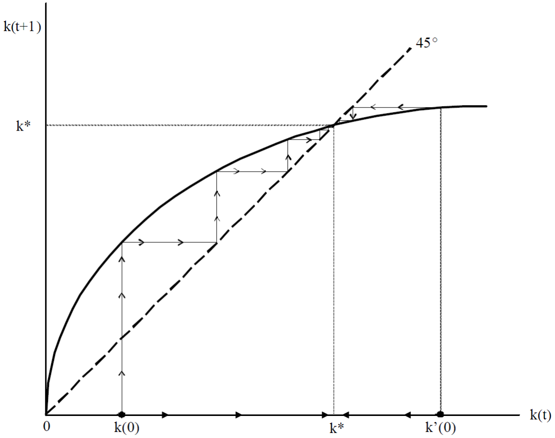
Log CD case: Consumption, savings and capital accumulation

- And, finally, capital accumulation per worker becomes

$$
\begin{aligned}
k_{t+1}(1+n) &= s_t \\
&= \frac{1}{2+\rho} w_t \\
&= \frac{1}{2+\rho}(1-\alpha)k_t^{\alpha}
\end{aligned}
$$

- k_{t+1} is a concave function of k_t, so there will be a unique steady state, and we'll converge to it

Dynamics



- Could analyze policy experiments - fall in ρ?

Dynamics

- So this looks a lot like Solow, there is nothing new in the aggregate dynamics! Constant savings rate, convergence to BGP along which every per-worker variable is constant
- It turns out that things become very different very quickly when we depart from logCD assumption...

Outline

Welfare

- The one major difference between the BGPs of Ramsey and Diamond's models: Welfare
- In Ramsey, it is not Pareto optimal to accumulate capital above the MGR (MGR = SS = BGP i.e., variables denoted with a "*")
- Ramsey households don't accumulate above the BGP in the CE (competitive or decentralized equilibrium) because the FWT holds: The CE is Pareto optimal
- Different in OLG! Households may accumulate capital above the BGP, as this is not Pareto optimal

Golden Rule in OLG

- Let's calculate the GR level in the OLG economy.
- As usual, the GR maximizes BGP consumption such that $c = f(k) - nk$. Thus:

$$f'(k^{gr}) = n$$

- Using the capital accumulation equation on the BGP, we get

$$k^* = \left[\frac{1-\alpha}{(1+n)(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

and, thus

$$f'(k^*) = \alpha\,(k^*)^{\alpha-1} = \frac{\alpha}{1-\alpha}(1+n)(2+\rho)$$

Golden Rule in OLG

- So, capital is above the GR if:

$$k^* > k^{gr} \iff f'(k^*) < n = f'(k^{gr}) \iff \frac{\alpha}{1-\alpha}(1+n)(2+\rho) < n$$

- No reason why that shouldn't occur, depends on parameters
- Capital is more likely to be above consumption-maximizing level in the long run if
  - Agents are relatively patient (low ρ)
  - Returns to capital don't diminish too fast (low capital income share, α)
  - Population growth?

Social Planner in OLG with $k^* > k^{gr}$

- To understand
  1. Why capital above GR is not optimal and
  2. Why this can happen in the competitive equilibrium of the OLG model

  let's consider what the Social Planner would do

Social Planner in OLG with $k^* > k^{gr}$

- To see why it is inefficient to have $k^* > k^{gr}$, assume to introduce a social planner into a OLG economy at the BGP
- If the planner does nothing to alter capital per worker, $c^* = f(k^*) - nk^*$
- Assume a one-off reduction in investment at period T ($\Delta k < 0$) to sustain higher consumption, then move straight to the new BGP
- Clearly feasible. Are we also better off?

Social Planner in OLG with $k^* > k^{gr}$

- What is the change in aggregate consumption per worker?
- In T the resources available for consumption are:

$$c_T = f(k^*) + (k^* - k^{gr}) - nk^{gr}$$

- For $\forall t > T$:

$$c^{gr} = f(k^{gr}) - nk^{gr}$$

- So $\Delta c > 0, \forall t$!

Social Planner in OLG with $k^* > k^{gr}$

- The Planner essentially takes savings from the current young and distributes it across generations as consumption
- The young are happy with this because they are promised a consumption transfer when old that is higher than what they give up today
  - While savings have a gross return $1 + r_t$, the planner's transfer has an implicit return of $1 + n$ (since there are $1 + n$ times as many young as old)
  - So this is a good deal for the current young when $n > r_t$, that is when $k^* > k^{gr}$
- Future generations are clearly better off: They enjoy higher consumption
- Everyone consumes more and is better off

Welfare in OLG

- Why don't agents in the OLG CE do the same thing as the Social Planner when $k^* > k^{gr}$?
- Recall the Social Planner's transfer scheme: He takes savings from the current young and **promises them a transfer when they are old**
- This is not implementable in the CE: Current young would have to enter into a contract with tomorrow's unborn young
- The underlying reason for the lack of Pareto optimality when $k^* > k^{gr}$ thus is that markets are incomplete (one of the FWT assumptions is broken!)
- Which assumption can we introduce to restore Pareto optimality in an OLG model?

Summary: Welfare in OLG

- Competitive equilibrium of OLG economy may feature capital above the Golden Rule
  - Because there are no restrictions on how patient agents are - they only live for a finite time
- If the economy goes on forever, then this is not Pareto optimal