Problem Set 1

Macroeconomics III

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Introduction

- Before exercise classes
  - Read the problem set
  - Give the problems a shot or at least think about possible approaches
- During
  - Participate in class
  - Stay focused and write down the solutions - Problem sets are often quite long compared to the length of the exercise classes
- After
  - Go through the solutions
  - Come and see me if the solutions don't make sense to you

Elasticities

The elasticity of $f(x)$ wrt. x is the percentage change in $f(x)$ when x increases with one percent. We will typically use one of two approaches to calculate the elasticity:

- The definition:

$$\epsilon = \frac{df(x)}{dx}\frac{x}{f(x)} \tag{1}$$

  Compute the derivative and calculate

- The log-diff approach:

$$\epsilon = \frac{d \ln f(x)}{d \ln x} \tag{2}$$

  This method is often easier and faster

Question 1

Consider the instantaneous utility function

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}, \quad \theta > 0, \quad \theta \neq 1$$

Calculate the elasticity of marginal utility defined as:

$$\epsilon = -\frac{du'(c)}{dc}\frac{c}{u'(c)} = -u''(c)\frac{c}{u'(c)}$$

What does ϵ measure in this case?

- *"ϵ measures the percentage decrease in marginal utility when consumption increases by one percent."*

Problem 1 - solution

Solution: Taking derivatives of the utility function

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}$$

yields:

$$u'(c) = c^{-\theta} \quad , \quad u''(c) = -\theta c^{-\theta-1}$$

Plugging into the definition of ϵ

$$\epsilon = -u''(c)\frac{c}{u'(c)} = -\theta c^{-\theta-1}\frac{c}{c^{-\theta}} = \theta$$

Solving by the log-diff approach yields:

$$\ln u'(c) = -\theta \ln c \quad \Longrightarrow \quad \epsilon = -\frac{d\ln u'(c)}{d\ln c} = -\frac{d(-\theta \ln c)}{d\ln c} = \theta$$

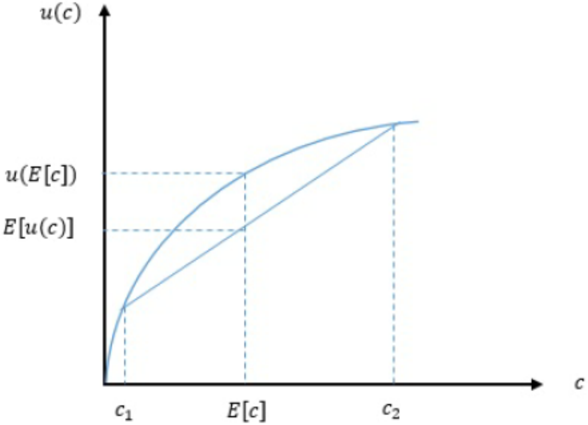
Hence, the elasticity of marginal utility is constant and equal to θ such that $\epsilon = \theta$.

Problem 1 - illustration

The elasticity of the marginal utility is related to risk aversion. A consumer is said to be risk-averse if the following holds:

$$u(E[c]) \geq E[u(c)]$$

The concavity of the utility function is governed by θ.



Problem 2

Calculate the elasticity of marginal utility of:

$$u(c) = \ln c$$

Use the log-diff approach:

Step 1: Take log of the derivate of the function:

$$\ln u'(c) = \ln \frac{d \ln c}{dc} = \ln \frac{1}{c} = -\ln c$$

Step 2: Insert into the formula:

$$\epsilon = -\frac{d \ln u'(c)}{d \ln c} = -\frac{d(-\ln c)}{d \ln c} = -(-1) = 1$$

Problem 3

Calculate and comment on:

$$\lim_{\theta \longrightarrow 1} \frac{c^{1-\theta} - 1}{1 - \theta}$$

L'Hopital's rule applies since:

$$\lim_{\theta \longrightarrow 1} c^{1-\theta} - 1 = 0 \qquad \text{and} \qquad \lim_{\theta \longrightarrow 1} 1 - \theta = 0$$

Using L'Hopital's rule and $\frac{da^x}{dx} = a^x \ln a$:

$$\lim_{\theta \longrightarrow 1} \frac{c^{1-\theta} - 1}{1 - \theta} = \lim_{\theta \longrightarrow 1} \frac{-c^{1-\theta} \ln c}{-1} = \ln c$$

Problem 6

σ is the elasticity of substitution. Suppose $\sigma = \frac{1}{\epsilon}$ holds. Does it make sense that a high elasticity of marginal utility implies a low elasticity of substitution?

Rewrite the MRS for CRRA utility:

$$\frac{p_1}{p_2} = MRS = \frac{u'(c_1)}{u'(c_2)} = \frac{c_1^{-\theta}}{c_2^{-\theta}} = \left(\frac{c_1}{c_2}\right)^{-\theta} = \left(\frac{c_2}{c_1}\right)^{\theta}$$

Taking logs of the MRS yields:

$$\ln MRS = \theta \ln \frac{c_2}{c_1}$$

Rewrite the elasticity of substitution and use the log of MRS:

$$\sigma = \frac{d(\frac{c_2}{c_1})\frac{p_1}{p_2}}{d(\frac{p_1}{p_2})\frac{c_2}{c_1}} = \frac{d(\frac{c_2}{c_1})MRS}{d(MRS)\frac{c_2}{c_1}} = \frac{d \ln \frac{c_2}{c_1}}{d \ln MRS} = \frac{d \ln \frac{c_2}{c_1}}{\theta d \ln \frac{c_2}{c_1}} = \frac{1}{\theta}$$

Where $\theta = \epsilon$ as shown in Problem 1.

Problem 6

Does it make intuitive sense that $\sigma = \frac{1}{\epsilon}$?

- If ϵ is very high, the marginal utility from good 1 or 2 will change quite a lot if the consumption bundle is changed. Therefore, a change in relative prices will will not change the consumption bundle a lot.
- Conversely, if the agent has a high elasticity of substitution, the agent will be willing to change the consumption bundle relatively more. This is rational and makes intuitive sense if the marginal utilities from consuming good 1 or 2 are not too sensitive to changes in the amount consumed.
- In the **Ramsey model** context, σ measures the preference for consumption smoothing across periods. **CRRA utility functions** are then useful because they imply a constant σ.

Problem 10

Production function with constant returns to scale (such as Cobb-Douglas):

$$F(xK, xL) = xF(K, L), \quad \forall x > 0$$

Show that

$$\frac{1}{L} F(K, L) = f(k)$$

where $k = \frac{K}{L}$ and $f(k) = F(k, 1)$.
Show that

$$\frac{\partial F(K, L)}{\partial K} = f'(k)$$

And

$$\frac{\partial F(K, L)}{\partial L} = f(k) - kf'(k)$$

Problem 10

Show that $\frac{1}{L}F(K,L) = f(k)$**.**

$$\frac{1}{L}F(K,L) = F(\frac{K}{L}, \frac{L}{L}) = F(k,1) = f(k)$$

Hence, $f(k)$ is the production per unit of labour. Since the economy exhibits constant returns to scale, the relevant metric for labour productivity is capital per labour unit.

Show that $\frac{\partial F(K,L)}{\partial K} = f'(k)$**.** Rewrite $F(K,L)$ and take the derivative.

$$\frac{\partial F(K,L)}{\partial K} = \frac{\partial (Lf(k))}{\partial K} = L\frac{\partial f(\frac{K}{L})}{\partial K} = L \cdot f'(k)\frac{1}{L} = f'(k)$$

Hence, production increases with $f'(k)$, which is the marginal product of capital per unit of labour, as capital increases.

Problem 10

Show that $\frac{\partial F(K,L)}{\partial L} = f(k) - kf'(k)$.

- Start by taking the derivative of $f(k)$ with regards to L.

$$\frac{\partial F(K,L)}{\partial L} = \frac{\partial Lf(k)}{\partial L} = \frac{\partial L}{\partial L} f(k) + L\frac{\partial f(k)}{\partial L} = f(k) + L\frac{\partial f(k)}{\partial L}$$

- Find $L\frac{\partial f(k)}{\partial L}$:

$$L\frac{\partial f(k)}{\partial L} = Lf'(k)\frac{\partial k}{\partial L} = Lf'(k)\frac{\partial(\frac{K}{L})}{\partial L} = Lf'(k)(-\frac{K}{L^2}) = -f'(k)k$$

- Insert into the derivative of $F(k, L)$ wrt. L:

$$\frac{\partial F(K,L)}{\partial L} = f(k) + L\frac{\partial f(k)}{\partial L} = f(k) - f'(k)k$$

There is a positive effect of $f(k)$ from adding one unit of labour, since $f(k)$ is the production per labour unit. Production does, however, also decrease with $f'(k)k$ since capital must be shared between more labour.

Problem 12

The household's problem is given by:

$$\max_{\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\underbrace{a_t R_t + w_t - a_{t+1}}_{=c_t}), \text{ s.t. } a_0 \text{ given, } \lim_{T\to\infty} q_T a_{T+1} \geq 0$$

Let $\hat{a} \equiv \{\hat{a}_{t+1}\}_{t=0}^{\infty}$ denote a plan that satisfies the Euler equation at all times as well as $\lim_{T\to\infty} q_T \hat{a}_{T+1} = 0$. Let $\bar{a} \equiv \{\bar{a}_{t+1}\}_{t=0}^{\infty}$ denote an alternative plan that satisfies $\lim_{T\to\infty} q_T \bar{a}_{T+1} \geq 0$. We want to show that the former dominates the latter.

Dynamic budget constraint:

$$c_t + a_{t+1} = a_t R_t + w_t$$

Prices:

$$q_t = \frac{1}{R_1 R_2 \ldots R_t}$$

Constraints

Two important constraints:

No-Ponzi-Game condition (NPGC):	$\lim_{T\to\infty} q_T a_{T+1} \geq 0$
Transversality Condition (TVC):	$\lim_{T\to\infty} q_T a_{T+1} = 0$

NPGC: A constraint that prevents households from rolling over debt forever without serving it - agents cannot finance interest payments by taking up new loans.

Euler condition: Solve the household's problem:

$$\beta^t u'(c_t) = \beta^{t+1} R_{t+1} u'(c_{t+1})$$
$$\frac{\partial u_t}{\partial c_t} = \beta R_{t+1} \frac{\partial u_{t+1}}{\partial c_{t+1}}$$

Problem 12 - Quick solution (1/3)

Assume the condition is:

$$\lim_{T\to\infty} q_T a_{T+1} = k \geq 0$$

Where households choose $k \geq 0$. Use the budget constraint: $c_t + a_{t+1} = a_t R_t + w_t$ and $q_t R_t = q_{t-1}$.

Combine these equations:

$$\begin{aligned} c_t - w_t &= a_t R_t - a_{t+1} \\ q_t(c_t - w_t) &= q_t(a_t R_t - a_{t+1}) = q_{t-1} a_t - q_t a_{t+1} \end{aligned}$$

Iterating yields:

$$\begin{aligned} \textstyle\sum_{t=0}^{T} q_t(c_t - w_t) =& (a_0 R_0 - q_0 a_1) + (q_0 a_1 - q_1 a_2) \\ & + (q_1 a_2 - q_2 a_3) + (q_2 a_3 - q_3 a_4) \\ & \dots \\ & + (q_{T-2} a_{T-1} - q_{T-1} a_T) + (q_{T-1} a_T - q_T a_{T+1}) \\ & = a_0 R_0 - q_T a_{T+1} \end{aligned}$$

Problem 12 - Quick solution (2/3)

The intertemporal budget constraint is found by taking limits:

$$\sum_{t=0}^{T} q_t(c_t - w_t) = a_0 R_0 - \lim_{T\to\infty} q_T a_{T+1} = a_0 R_0 - k$$

Taking derivatives with regards to k:

$$\sum_{t=0}^{T} q_t \frac{\partial c_t}{\partial k} = -1$$

Hence, we know that the overall effect of increasing k is lower consumption. Next turn to the Euler equation:

$$\frac{\partial u_t}{\partial c_t} = \beta R_{t+1} \frac{\partial u_{t+1}}{\partial c_{t+1}}$$

Differentiating wrt. k:

$$u_t'' \frac{\partial c_t}{\partial k} = \beta R_{t+1} u_t'' \frac{\partial c_{t+1}}{\partial k} \quad \Longrightarrow \quad sign\Big(\frac{\partial c_t}{\partial k}\Big) = sign\Big(\frac{\partial c_{t+1}}{\partial k}\Big)$$

Hence, we know that the consumption will decrease at every point in time from an increase in k.

Problem 12 - Quick solution (3/3)

- $\hat{a} \equiv \{\hat{a}_{t+1}\}_{t=0}^{\infty}$ satisfies $\lim_{T\to\infty} q_T \hat{a}_{T+1} = 0$.
- $\bar{a} \equiv \{\bar{a}_{t+1}\}_{t=0}^{\infty}$ satisfies $\lim_{T\to\infty} q_T \bar{a}_{T+1} \geq 0$.

The households will choose to minimize $\lim_{T\to\infty} q_T \hat{a}_{T+1}$, which implies that $\hat{a}$ weakly dominates $\bar{a}$.

Intuitively, households will choose to consume their wealth rather than have net positive wealth in the infinite horizon.

Problem 12 - Alternative approach (1/5)

Recall that a first order Taylor expansion for a function with two variables around (x_0, y_0) is given by:

$$\tilde{f}(x,y) = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y - y_0)$$

Step 1: We consider utility as a function of assets, such that:

$$u_t = u(c_t) = u(a_t R_t + w_t - a_{t+1})$$

Compute the first order Taylor expansion of u_t around $(\hat{a}_t, \hat{a}_{t+1})$:

$$\tilde{u}_t = \hat{u}_t + R_t \hat{u}_t'(a_t - \hat{a}_t) - \hat{u}_t'(a_{t+1} - \hat{a}_{t+1})$$

We define the aproximation error as

$$\tilde{u}_t - u_t$$

Since u_t is concave, we know that the minimum of the approximation error is at $\tilde{u}_t = u_t$. Think about a tangent linear line around a point x_0 on a concave curve - the linear line will be above the concave curve at all points except x_0. Hence, $\tilde{u}_t \geq u_t$.

Problem 12 - Alternative approach (2/5)

We know that $\tilde{u}_t - u_t \geq 0$, which leads us to:

$$\tilde{u}_t - u_t = \hat{u}_t + R_t \hat{u}'_t(a_t - \hat{a}_t) - \hat{u}'_t(a_{t+1} - \hat{a}_{t+1}) - u_t \geq 0$$

Move u_t and $\hat{u}_t$ to the right hand side:

$$R_t \hat{u}'_t(a_t - \hat{a}_t) - \hat{u}'_t(a_{t+1} - \hat{a}_{t+1}) > u_t - \hat{u}_t, \quad \text{for } (a_t, a_{t+1}) \neq (\hat{a}_t, \hat{a}_{t+1})$$

Step 2: Consider the two solutions, $\bar{a}_t$ and $\hat{a}_t$, and the sum of approximation errors.

$$\sum_{t=0}^{T} \beta^t \hat{u}'_t \Big(R_t(\bar{a}_t - \hat{a}_t) - (\bar{a}_{t+1} - \hat{a}_{t+1}) \Big) \geq \sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t)$$

The right hand side is the present value of the sum of utility for plan $\bar{a}$ minus the present value of the sum of utility for plan $\hat{a}$. If the sum is above zero, plan $\bar{a}$ is preferred and if it is less than zero plan $\hat{a}$ is preferred. If it is zero, the household is indifferent.

Step 3: Simplify the left-hand side sums. Using the Euler equation, we write out the sum (for $\hat{a}_t$):

$$\begin{aligned}
\sum_{t=0}^{T} \beta^t \hat{u}'_t(-R_t \hat{a}_t + \hat{a}_{t+1}) = &- \hat{u}'_0 R_0 \hat{a}_0 + \hat{u}'_0 \hat{a}_1 \\
&- \beta \hat{u}'_1 R_1 \hat{a}_1 + \beta \hat{u}'_1 \hat{a}_2 \\
&- \beta^2 \hat{u}'_2 R_2 \hat{a}_2 + \beta^2 \hat{u}'_1 \hat{a}_2 \\
&\dots \\
&- \beta^{T-1} \beta \hat{u}'_{T-1} R_{T-1} \hat{a}_{T-1} + \beta^{T-1} \hat{u}'_{T-1} \hat{a}_T \\
&- \beta^T \hat{u}'_T R_T \hat{a}_T + \beta^T \hat{u}'_T \hat{a}_{T+1} \\
= &- \hat{u}'_0 R_0 \hat{a}_0 + \beta^T \hat{u}'_T \hat{a}_{T+1}
\end{aligned}$$

Most of the terms cancel due to the Euler equation:

$$\beta \hat{u}'_1 = \beta^2 \hat{u}'_2 R_2$$

Problem 12 - Alternative approach (4/5)

We know that:

$$\sum_{t=0}^{T} \beta^t \hat{u}_t'(-R_t \hat{a}_t + \hat{a}_{t+1}) = -\hat{u}_0' R_0 \hat{a}_0 + \beta^T \hat{u}_T' \hat{a}_{T+1}$$

Since a_0 will be the same for both $\bar{a}$ and $\hat{a}$, we get:

$$\sum_{t=0}^{T} \beta^t \hat{u}_t' \Big(R_t(\bar{a}_t - \hat{a}_t) - (\bar{a}_{t+1} - \hat{a}_{t+1}) \Big) \geq \sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t)$$

$$\beta^T \hat{u}_T' (\hat{a}_{T+1} - \bar{a}_{T+1}) \geq \sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t)$$

Step 4: Next, substitute with prices given as:

$$\beta^T \hat{u}_t' = q_T \hat{u}_0'$$

Hence, we get:

$$u_0'(q_T \hat{a}_{T+1} - q_T \bar{a}_{T+1}) \geq \sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t)$$

Problem 12 - Alternative approach (5/5)

Taking limits:

$$0 \geq \underbrace{u_0'}_{>0}\big(\underbrace{\lim_{T\to\infty} q_T \hat{a}_{T+1}}_{=0 \text{ by TVC}} - \underbrace{\lim_{T\to\infty} q_T \bar{a}_{T+1}}_{\geq 0 \text{ by NPGC}}\big) \geq \sum_{t=0}^{\infty} \beta^t (\bar{u}_t - \hat{u}_t)$$

The left hand side is negative, which implies that the right hand side is negative as well. Thus, the discounted utility for $\hat{a}$ is greater or equal to the discounted utility of $\bar{a}$.

Hence, the TVC weakly dominates the NPGC. It is not strictly dominant since satisfying the NPGC does not rule out satisfying the TVC.