Microeconomics III: Problem Set 6[a]

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[a] Slides created for exercise class with reservation for possible errors.

Outline

PS6, Ex. 1 (A): Sequential bargaining

PS6, Ex. 2 (A): Infinite-horizon bargaining

PS6, Ex. 3: Dynamic games (imperfect information)

PS6, Ex. 4: Infinite-horizon bargaining with different discount factors

PS6, Ex. 5: Cournot, colluding to every-ones benefit?

PS6, Ex. 1 (A): Sequential bargaining

PS6, Ex. 1 (A): Sequential bargaining

Consider the sequential bargaining game discussed in Lecture 6, but now with $K \geq 1$ stages (where K is some arbitrary but fixed integer). Suppose $\delta = 1$ and $K = 1, 2, 3$. Is there a first-mover advantage? Does your answer depend on the value of K?

PS6, Ex. 1 (A): Sequential bargaining

Explain δ mathematically

delta is the discount factor which the payoff in the next game will be multiplied by, so if there player stand to gain 1 in the next stage, and $\delta = 0.5$, it is only worth $1 * 0.5 = 0.5$ to the player in the current stage.

Explain δ intuitively

Intuitively δ is the factor showing how patient the players are. The higher δ, thee less the players will mind waiting for the next stage.

Explain the case $\delta = 0$

In the case $\delta = 0$, the players will have their payoff multiplied by 0 in the next stage, so the game stages into an ultimatum game where the first mover can offer the other player anything and they will accept. There is a first mover advantage.

Explain the case $\delta = 1$

In the case $\delta = 1$, the players will have their payoff multiplied by 1 in the next stage, so they won't care whether the game goes for another astage. This will be the case for each stage until the final stage, which will then be an ultimatum game where the last mover can offer the other player anything and they will accept. There is no first mover advantage, but there is a last mover advantage.

Explain whether it depends on K

For δ=1, the last mover will get the whole price pool, no matter how many stages (K) the game is. The only case with a first mover advantage is for $K = 1$, in which the first move is the same as the last.

PS6, Ex. 2 (A): Infinite-horizon bargaining

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Question 2.3 from Gibbons (p.131) looks at the infinite-horizon bargaining game where player 1 has discount factor δ_1 and player 2 has discount factor δ_2. It shows that the backward-induction outcome of this game is

$$\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right) \tag{1}$$

Discuss how these payoffs change as each player becomes more or less patient, i.e. as we vary δ_1 and δ_2. What is the intuition? Show that these payoffs simplify to those derived in Lecture 6

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right) \tag{2}$$

for the case where $\delta_1 = \delta_2$

PS6, Ex. 2 (A): Infinite-horizon bargaining

Part one: For the payoffs: $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ Discuss how the payoff change as each player becomes more or less patient.

(Step a) Write up partial derivatives for δ_2's and δ_1's effect on the outcome for player 1, are the partial derivatives positive or negative?

Information so far:

PS6, Ex. 2 (A): Infinite-horizon bargaining

Part one: For the payoffs: $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ Discuss how the payoff change as each player becomes more or less patient.

(Step a) Write up partial derivatives for δ_2's and δ_1's effect on the outcome for player 1, are the partial derivatives positive or negative?

(Step b) Use the fact that it's a zero sum game to look at the change in outcome for player 2

Information so far:

1. $\frac{\partial s_1*}{\partial \delta_1} = \frac{(1-\delta_2)\delta_2}{(1-\delta_1\delta_2)^2} > 0$
2. $\frac{\partial s_1*}{\partial \delta_2} = -\frac{1-\delta_1}{(1-\delta_1\delta_2)^2} < 0$

PS6, Ex. 2 (A): Infinite-horizon bargaining

Part one: For the payoffs: $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ Discuss how the payoff change as each player becomes more or less patient.

(Step a) Write up partial derivatives for δ_2's and δ_1's effect on the outcome for player 1, are the partial derivatives positive or negative?

(Step b) Use the fact that it's a zero sum game to look at the change in outcome for player 2

Answer Player 1s payoff is increasing in δ_1 and decreasing in δ_2, vice versa for Player 2. This intuitively makes sense, because player i's bargaining power in later stages will increase when his patience increase relative to player j.

Information so far:

1 $\frac{\partial s_1*}{\partial \delta_1} = \frac{(1-\delta_2)\delta_2}{(1-\delta_1\delta_2)^2} > 0$

2 $\frac{\partial s_1*}{\partial \delta_2} = -\frac{1-\delta_1}{(1-\delta_1\delta_2)^2} < 0$

PS6, Ex. 2 (A): Infinite-horizon bargaining

Part two: For the payoffs: $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ show that for $\delta_2 = \delta_1$ the payoffs simplify to $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$

Write up the payoffs with $\delta = \delta_1 = \delta_2$ and use that: $1 - x^2 = (1+x)(1-x)$, to simplify

$$\left(\frac{1-\delta}{1-\delta^2}, \frac{\delta(1-\delta)}{1-\delta^2}\right) \Rightarrow \left(\frac{1-\delta}{(1-\delta)(1+\delta)}, \frac{\delta(1-\delta)}{(1-\delta)(1+\delta)}\right) \Rightarrow \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$$

PS6, Ex. 3: Dynamic games (imperfect information)

PS6, Ex. 3: Dynamic games (imperfect information)

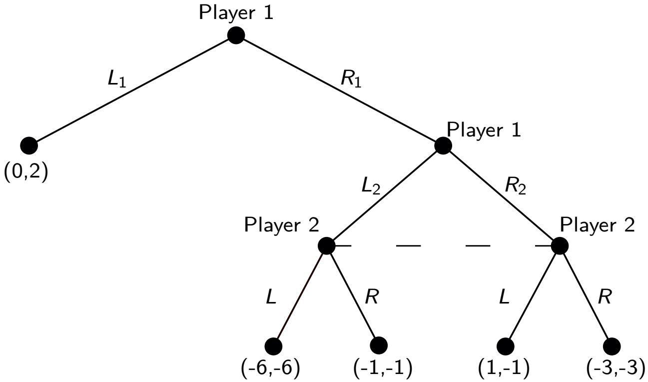
Find the SPNE in the four games.

Hints:

1. It becomes much easier to grasp dynamic games with imperfect information if you write the part with imperfect information in normal form (bi-matrix).
2. Be careful to cover all of the strategy profile (in every subgame!) when writing up the subgame perfect Nash Equilibria (SPNE).

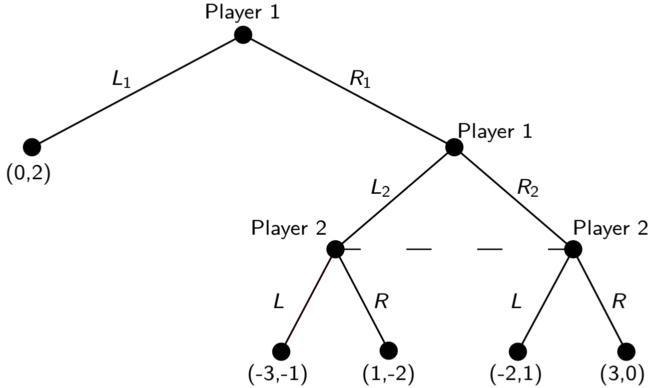
PS6, Ex. 3.a: Dynamic games (imperfect information)

(a) Find the SPNE in the following game:



PS6, Ex. 3.a: Dynamic games (imperfect information)

(a) Find the SPNE in the following game:



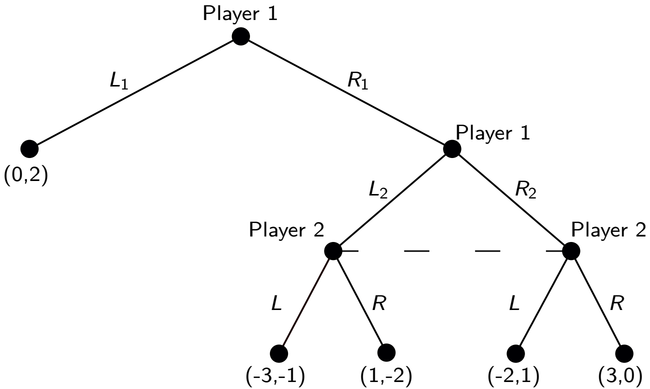
2^{nd} and 3^{rd} stage in normal form:

Player 2

Player 1	L	R
L_2	-3, -1	1, -2
R_2	-2, 1	3, 0

PS6, Ex. 3.a: Dynamic games (imperfect information)

(a) Find the SPNE in the following game:



2^{nd} and 3^{rd} stage in normal form:

		Player 2	
		L	R
Player 1	L_2	-3, -1	1, -2
	R_2	-2, 1	3, 0

PS6, Ex. 3.a: Dynamic games (imperfect information)

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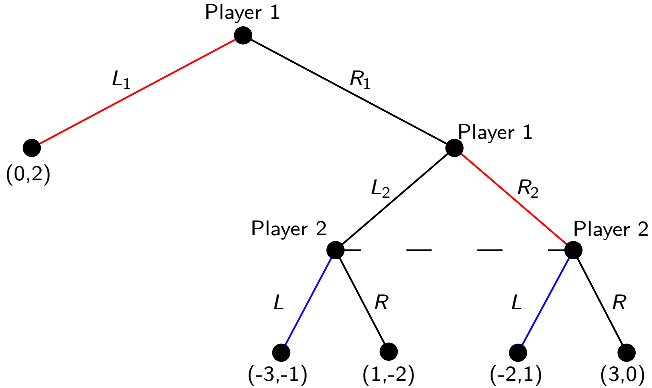


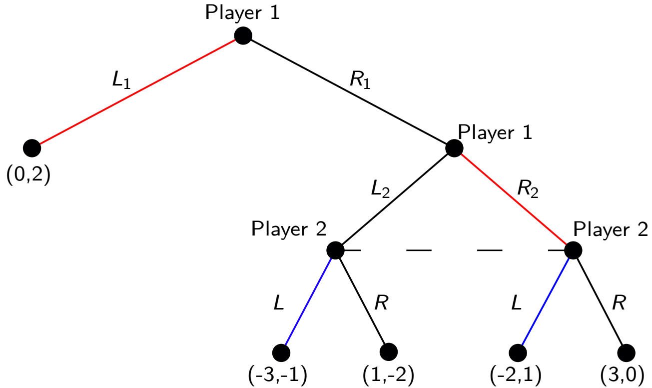
2nd and 3rd stage in normal form:

		Player 2	
		L	R
Player 1	L_2	-3, -1	1, -2
	R_2	-2, 1	3, 0

Write up the SPNE!

PS6, Ex. 3.a: Dynamic games (imperfect information)

(a) Find the SPNE in the following game:



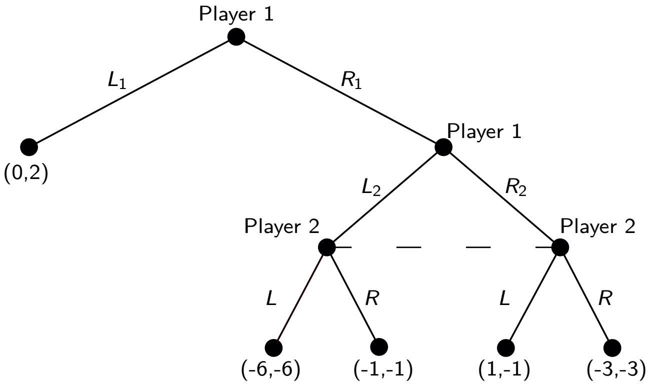
2nd and 3rd stage in normal form:

Player 1		Player 2: L	R
	L_2	-3, -1	1, -2
	R_2	-2, 1	3, 0

$SPNE = \{s_1^*, s_2^*\} = \{(L_1, R_2), L\}$ with outcome (0,2).

PS6, Ex. 3.b: Dynamic games (imperfect information)

(b) Find the SPNE in the following game:



PS6, Ex. 3.b: Dynamic games (imperfect information)

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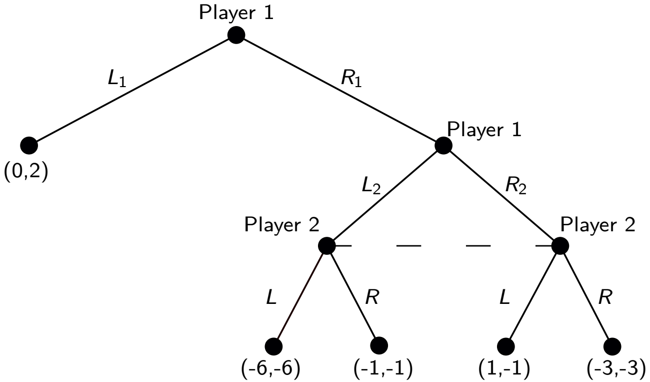
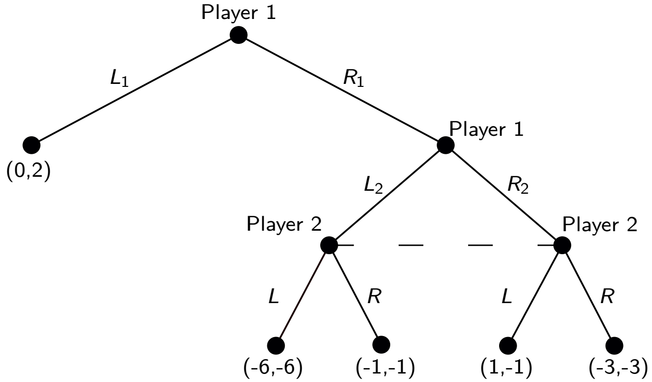


2^{nd} and 3^{rd} stage in normal form:

		Player 2	
		L	R
Player 1	L_2	-6, -6	-1, -1
	R_2	1, -1	-3, -3

PS6, Ex. 3.b: Dynamic games (imperfect information)

(b) Find the SPNE in the following game:



2nd and 3rd stage in normal form:

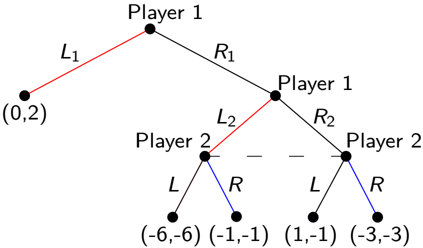
		Player 2	
		L	R
Player 1	L_2	-6, -6	-1, -1
	R_2	1, -1	-3, -3

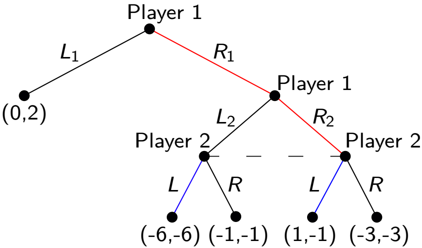
Two different pure strategy NE (PSNE) in the subgame. What now?

PS6, Ex. 3.b: Dynamic games (imperfect information)

(b) Find the SPNE in the following game:

we have two subgame perfect solutions:



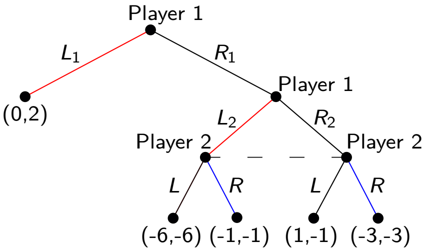


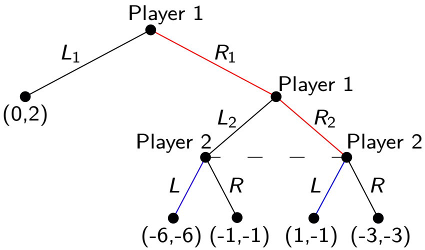
Write up the SPNE!

PS6, Ex. 3.b: Dynamic games (imperfect information)

(b) Find the SPNE in the following game:

we have two subgame perfect solutions:

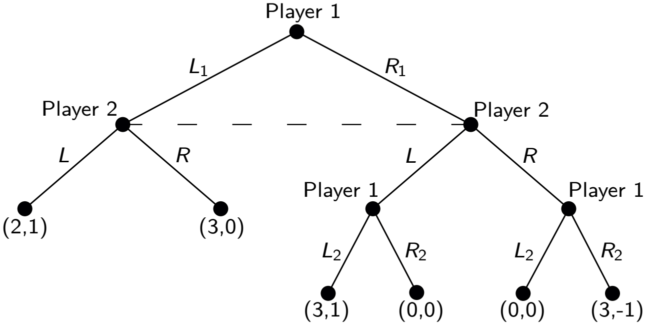




$SPNE = \{s_1^*, s_2^*\} = \{(L_1, L_2), R; (R_1, R_2), L\}$ with outcomes {(0,2);(1,-1)}.

PS6, Ex. 3.c: Dynamic games (imperfect information)

(c) Find the SPNE in the following game:



PS6, Ex. 3.c: Dynamic games (imperfect information)

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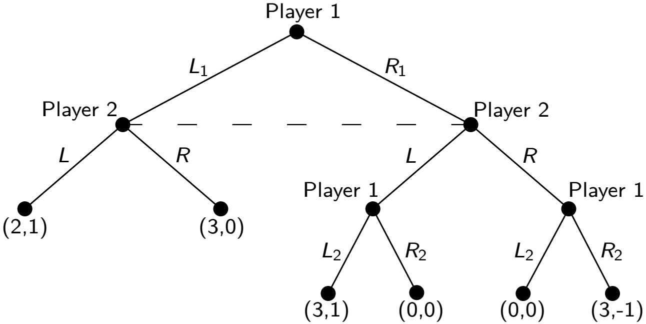
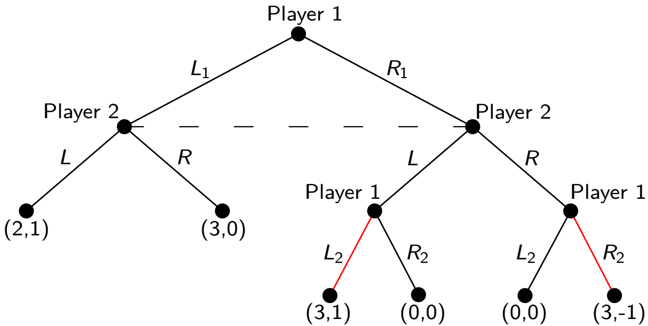


Backwards Induction: First solve the 3^{rd} stage.

PS6, Ex. 3.c: Dynamic games (imperfect information)

(c) Find the SPNE in the following game:



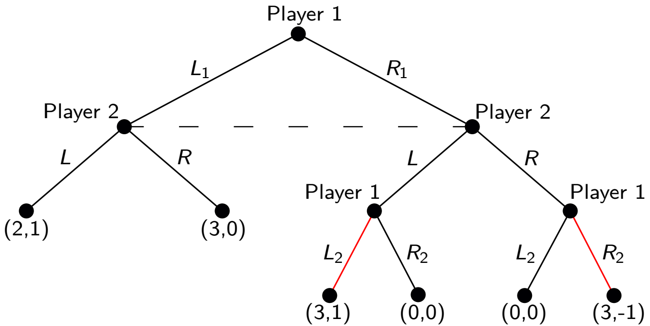
Backwards Induction: First solve the 3$^{\text{rd}}$ stage.

1$^{\text{st}}$ and 2$^{\text{nd}}$ stage in normal form (taking the 3$^{\text{rd}}$ stage as given):

Player 1		Player 2: L	Player 2: R
	L_1	2, 1	3, 0
	R_1	3, 1	3, -1

PS6, Ex. 3.c: Dynamic games (imperfect information)

(c) Find the SPNE in the following game:



Backwards Induction: First solve the 3rd stage.

1st and 2nd stage in normal form (taking the 3rd stage as given):

	Player 2 L	Player 2 R
Player 1 L_1	2, 1	3, 0
Player 1 R_1	3, 1	3, -1

PS6, Ex. 3.c: Dynamic games (imperfect information)

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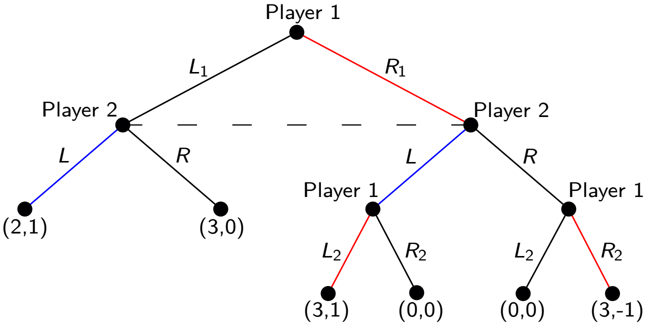


Backwards Induction: First solve the 3rd stage.

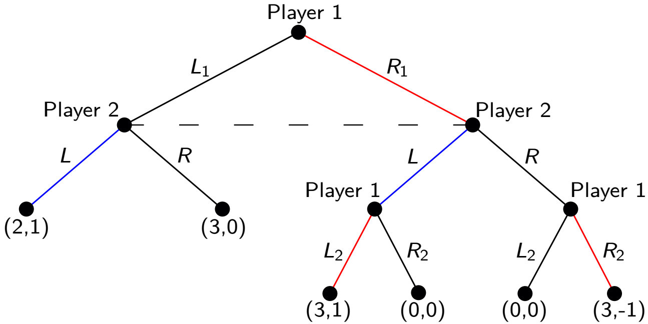
1st and 2nd stage in normal form (taking the 3rd stage as given):

		Player 2	
		L	R
Player 1	L_1	2, 1	3, 0
	R_1	3, 1	3, -1

Consider how many subgames there are and write up the SPNE.

PS6, Ex. 3.c: Dynamic games (imperfect information)

(c) Find the SPNE in the following game:



Backwards Induction: First solve the 3rd stage.

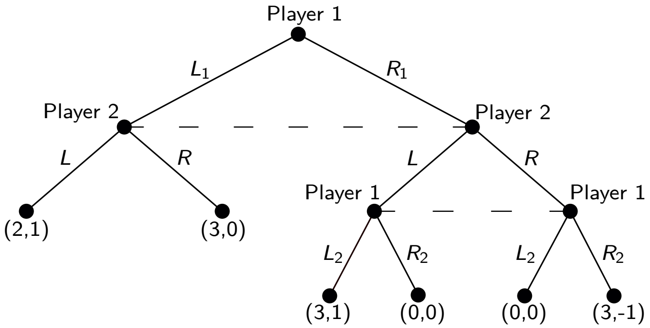
1st and 2nd stage in normal form (taking the 3rd stage as given):

		Player 2	
		L	R
Player 1	L_1	2, 1	3, 0
	R_1	3, 1	3, -1

$SPNE = \{s_1^*, s_2^*\} = \{(R_1, L_2, R_2), L\}$ with outcome (3,1).

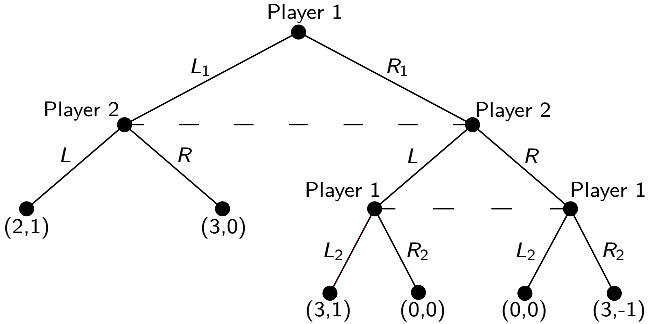
PS6, Ex. 3.d: Dynamic games (imperfect information)

(d) Find the SPNE in the following game:

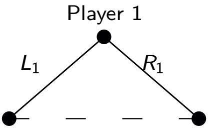


PS6, Ex. 3.d: Dynamic games (imperfect information)

(d) Find the SPNE in the following game:



2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):

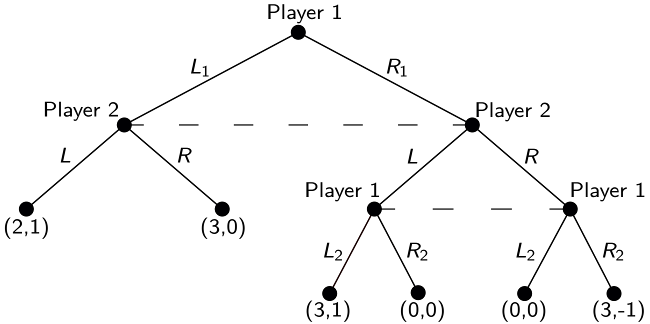


	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

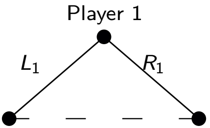
	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

PS6, Ex. 3.d: Dynamic games (imperfect information)

(d) Find the SPNE in the following game:



2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):



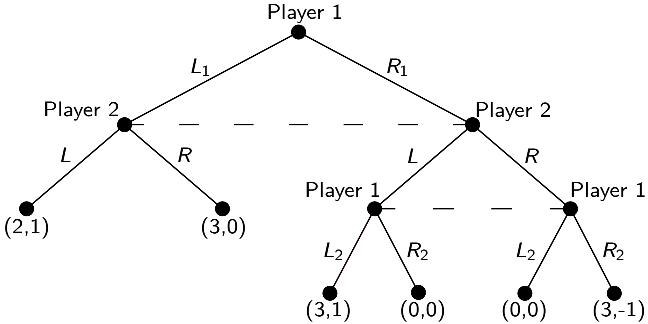
	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

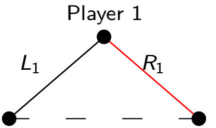
Which equilibria are subgame perfect?

PS6, Ex. 3.d: Dynamic games (imperfect information)

(d) Find the SPNE in the following game:



2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):



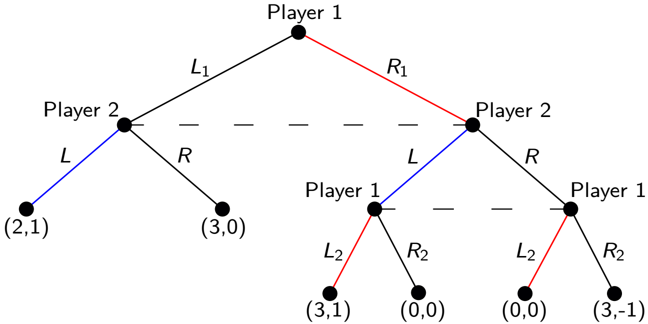
	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

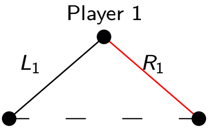
Player 2: R is strictly dominated by L. Player 1: Expecting L, she plays (R_1, L_2).

PS6, Ex. 3.d: Dynamic games (imperfect information)

(d) Find the SPNE in the following game:



2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):



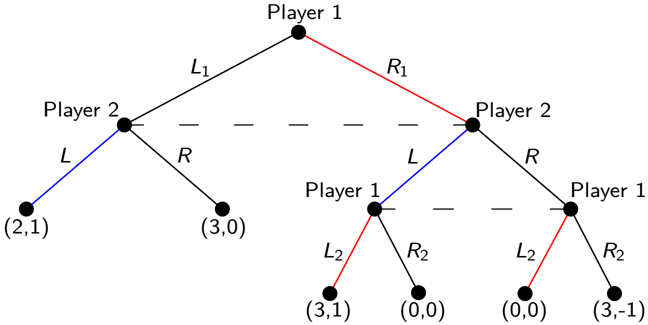
	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

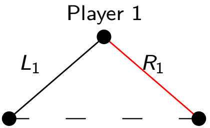
Write up the SPNE!

PS6, Ex. 3.d: Dynamic games (imperfect information)

(d) Find the SPNE in the following game:



2^{nd} and 3^{rd} stage in normal form (Player 1 knows her own action in 1^{st} stage):



	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

$SPNE = \{s_1^*, s_2^*\} = \{(R_1, L_2), L\}$ with outcome (3,1).

PS6, Ex. 4: Infinite-horizon bargaining with different discount factors

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Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement

$$(s^*, 1 - s^*) = \left(\frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \frac{\delta_2 (1 - \delta_1)}{1 - \delta_1 \delta_2} \right) \tag{3}$$

which player 2 accepts.

PS6, Ex. 4: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement

$$(s^*, 1 - s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right) \tag{4}$$

which player 2 accepts.

Hints:

1. Start with a three stage game where Player 1 gets a payoff s_3 in stage 3.
2. Use this to find a stationary solution where $s_1 = s_3$.
3. Remember that the outcome in period $\boldsymbol{t}$ is always denoted $\boldsymbol{s_t}$ for Player 1 and $\boldsymbol{1 - s_t}$ for Player 2.

PS6, Ex. 4 (a): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1 - s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step a) Start with a three stage game where Player 1's payoff in stage 3 is denoted s_3. Write up the outcome for Player 1 in stage 3. Then use the potential outcome of stage 3 to find the outcome in stage 2. Do the same for stage 1 with respects to stage 2.

Stage 3 What does Player 1 propose? Does Player 2 accept? What does Player 1 get himself?

PS6, Ex. 4 (a): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step a) Start with a three stage game where Player 1's payoff in stage 3 is denoted s_3. Write up the outcome for Player 1 in stage 3. Then use the potential outcome of stage 3 to find the outcome in stage 2. Do the same for stage 1 with respects to stage 2.

Stage 3 P2 will choose to accept or decline an offer $1 - s_3 \in [0;1]$: She will accept anything. P1 proposes $1 - s_3$ which P2 accepts. P1 gets s_3 for himself.

Stage 2

PS6, Ex. 4 (a): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step a) Start with a three stage game where Player 1's payoff in stage 3 is denoted s_3. Write up the outcome for Player 1 in stage 3. Then use the potential outcome of stage 3 to find the outcome in stage 2. Do the same for stage 1 with respects to stage 2.

Stage 3 P2 will choose to accept or decline an offer $1 - s_3 \in [0; 1]$: She will accept anything. P1 proposes $1 - s_3$ which P2 accepts. P1 gets s_3 for himself.

Stage 2 P1 will choose to accept or decline an offer $s_2 \in [0; 1]$: He will accept if $s_2 \geq s_3\delta_1$. P2 proposes $s_2 = s_3\delta_1$ which P1 accepts. P2 gets $1 - s_2 = 1 - s_3\delta_1$.

Stage 1

PS6, Ex. 4 (a): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1 - s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step a) Start with a three stage game where Player 1's payoff in stage 3 is denoted s_3. Write up the outcome for Player 1 in stage 3. Then use the potential outcome of stage 3 to find the outcome in stage 2. Do the same for stage 1 with respects to stage 2.

Stage 3 P2 will choose to accept or decline an offer $1 - s_3 \in [0; 1]$: She will accept anything. P1 proposes $1 - s_3$ which P2 accepts. P1 gets s_3 for himself.

Stage 2 P1 will choose to accept or decline an offer $s_2 \in [0; 1]$: He will accept if $s_2 \geq s_3\delta_1$. P2 proposes $s_2 = s_3\delta_1$ which P1 accepts. P2 gets $1 - s_2 = 1 - s_3\delta_1$.

Stage 1 P2 will choose to accept or decline an offer $1 - s_1 \in [0; 1]$: She will accept if $1 - s_1 \geq (1 - s_3\delta_1)\delta_2$. P1 proposes $1 - s_1 = (1 - s_3\delta_1)\delta_2$ which P2 accepts. P1 gets $s_1 = 1 - (1 - s_3\delta_1)\delta_2$.

PS6, Ex. 4 (b): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step b) Since the game is infinite, the players are playing the same game in stage 3 as in stage 1, thus, the outcome of stage 1 should be the same as in stage 3. Use this to find a stationary solution, where $s_1 = s_3$.

Outcomes for Player 1:
Stage 1: $s_1 = 1 - (1 - s_3\delta_1)\delta_2$.
Stage 3: s_3.

PS6, Ex. 4 (b): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step b) Since the game is infinite, the players are playing the same game in stage 3 as in stage 1, thus, the outcome of stage 1 should be the same as in stage 3. Use this to find a stationary solution, where $s_1 = s_3$.

Stationary solution:

$$s_1 = s_3 \Rightarrow$$

$$s^* = 1 - (1 - s^*\delta_1)\delta_2$$

Outcomes for Player 1:
Stage 1: $s_1 = 1 - (1 - s_3\delta_1)\delta_2$.
Stage 3: s_3.

PS6, Ex. 4 (b): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1 - s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step b) Since the game is infinite, the players are playing the same game in stage 3 as in stage 1, thus, the outcome of stage 1 should be the same as in stage 3. Use this to find a stationary solution, where $s_1 = s_3$.

Stationary solution:

$$\begin{aligned} s_1 &= s_3 \Rightarrow \\ s^* &= 1 - (1 - s^*\delta_1)\delta_2 \Rightarrow \\ s^* &= 1 - \delta_2 + s^*\delta_1\delta_2 \Rightarrow \\ s^*(1 - \delta_1\delta_2) &= 1 - \delta_2 \Rightarrow \\ s^* &= \frac{1 - \delta_2}{1 - \delta_1\delta_2} \end{aligned}$$

Outcomes for Player 1:
Stage 1: $s_1 = 1 - (1 - s_3\delta_1)\delta_2$.
Stage 3: s_3.

PS6, Ex. 4 (b): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step b) Since the game is infinite, the players are playing the same game in stage 3 as in stage 1, thus, the outcome of stage 1 should be the same as in stage 3. Use this to find a stationary solution, where $s_1 = s_3$.

Stationary solution:

$$\begin{aligned} s_1 &= s_3 \Rightarrow \\ s^* &= 1-(1-s^*\delta_1)\delta_2 \Rightarrow \\ s^* &= 1-\delta_2+s^*\delta_1\delta_2 \Rightarrow \\ s^*(1-\delta_1\delta_2) &= 1-\delta_2 \Rightarrow \\ s^* &= \frac{1-\delta_2}{1-\delta_1\delta_2} \end{aligned}$$

Insert in $1-s^*$, juggle a bit, and get:

$$(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

Outcomes for Player 1:
Stage 1: $s_1 = 1-(1-s_3\delta_1)\delta_2$.
Stage 3: s_3.

Bonus: Is there a first-mover advantage?

PS6, Ex. 4 (b): Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step b) Since the game is infinite, the players are playing the same game in stage 3 as in stage 1, thus, the outcome of stage 1 should be the same as in stage 3. Use this to find a stationary solution, where $s_1 = s_3$.

Stationary solution:

$$\begin{aligned} s_1 &= s_3 \Rightarrow \\ s^* &= 1-(1-s^*\delta_1)\delta_2 \Rightarrow \\ s^* &= 1-\delta_2+s^*\delta_1\delta_2 \Rightarrow \\ s^*(1-\delta_1\delta_2) &= 1-\delta_2 \Rightarrow \\ s^* &= \frac{1-\delta_2}{1-\delta_1\delta_2} \end{aligned}$$

Insert in $1-s^*$, juggle a bit, and get:

$$(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

Outcomes for Player 1:
Stage 1: $s_1 = 1-(1-s_3\delta_1)\delta_2$.
Stage 3: s_3.

P1 has first-mover advantage if:

$$\begin{aligned} 1-\delta_2 &\geq \delta_2(1-\delta_1) \Rightarrow \\ \frac{1}{\delta_2}-\frac{\delta_2}{\delta_2} &\geq 1-\delta_1 \Rightarrow \\ \frac{1}{\delta_2} &\geq 2-\delta_1 \Rightarrow \\ \delta_2 &\geq \frac{1}{2-\delta_1} \in \left[\frac{1}{2}, 1\right] \text{ for } \delta_1 \in [0,1] \end{aligned}$$

I.e. unless P2 is impatient (if δ_2 is low).

PS6, Ex. 5: Cournot, colluding to every-ones benefit?

PS6, Ex. 5: Cournot, colluding to everyone's benefit?

(A two stage game with simultaneous moves) On July 12, 2001, the presidents of Toyota and PSA Group, Fujio Cho and Jean-Martin Folz, decided to jointly develop a small city car This project was called B-Zero. The outcome of this project were Toyota Aygo, Peugeot 107, and Citroen C1 which are essentially differently named versions of the same car. So the firms entered into a collusive agreement at the R&D stage, but remained rivals in the final product market.

It is not surprising that the firms were not allowed to collude in the product market, as this would increase their monopoly power, which is costly for the consumers. But why was collusion allowed in the R&D market?

Below you are asked to show that if there are sufficient spillovers in R&D, collusion in R&D may be beneficial both for the firms and for the consumers.

PS6, Ex. 5: Cournot, colluding to everyone's benefit?

Consider an industry consisting of two firms. They face the inverse demand function given by

$$P(Q) = 2 - Q$$

where $Q = q_1 + q_2$ is the total quantity produced. Before production, each firm can engage in research activities that lower the cost of production for the entire industry. More precisely, the marginal cost of each firm is a function of the total amount of research undertaken by the two firms $(x1 + x2)$:

$$c = 1 - x_1 - x_2$$

Thus, each firm benefits from the research undertaken by the other firm. The cost of x_i units of research to firm i is given by

$$x_i^2$$

The timing of the game is as follows: In the first stage the firms choose the levels of research x_1 and x_2. In the second stage, after observing x_1 and x_2, the firms simultaneously and independently (i.e., as in a standard Cournot game) decide on the amounts of output (q_1 and q_2).

PS6, Ex. 5.a: Cournot, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Marginal cost of production: $c_{q1} = c_{q2} = c_q = 1 - x_1 - x_2$

3 Cost of research for firm i: $c_{xi} = x_i^2$

PS6, Ex. 5.a: Cournot, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step a) Write up the payoff function, taking research as given.

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Marginal cost of production: $c_{q1} = c_{q2} = c_q = 1 - x_1 - x_2$

3 Cost of research for firm i: $c_{xi} = x_i^2$

PS6, Ex. 5.a: Cournot, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step a) Write up the payoff function, taking research as given:

$$\begin{aligned}\pi_i &= P \cdot q_i - c_q \cdot q_i - c_{xi} \\ &= (P - c_q)q_i - c_{xi} \\ &= (2 - q_i - q_j - (1 - x_i - x_j))q_i - x_i^2 \\ &= (1 + x_i + x_j - q_i - q_j)q_i - x_i^2\end{aligned}$$

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Marginal cost of production: $c_{q1} = c_{q2} = c_q = 1 - x_1 - x_2$

3 Cost of research for firm i: $c_{xi} = x_i^2$

PS6, Ex. 5.a: Cournot, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step a) Write up the payoff function, taking research as given:

(Step b) Find $FOC_{q_i} : \delta\pi_i(q_i, q_j, x_i, x_j)/\delta q_i$.

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Marginal cost of production: $c_{q1} = c_{q2} = c_q = 1 - x_1 - x_2$

3 Cost of research for firm i: $c_{xi} = x_i^2$

4 $Payoff_i(q_i, q_j, x_i, x_j)$: $\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

PS6, Ex. 5.a: Cournot, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step a) Write up the payoff function, taking research as given.

(Step b) Find $FOC_{q_i} : \delta\pi_i(q_i, q_j, x_i, x_j)/\delta q_i$:

$$\frac{\delta\pi_i}{\delta q_i} = 1 - 2q_i - q_j + x_i + x_j = 0$$

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Marginal cost of production: $c_{q1} = c_{q2} = c_q = 1 - x_1 - x_2$

3 Cost of research for firm i: $c_{xi} = x_i^2$

4 $Payoff_i(q_i, q_j, x_i, x_j)$: $\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

PS6, Ex. 5.a: Cournot, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step a) Write up the payoff function, taking research as given.

(Step b) Find $FOC_{q_i} : \delta\pi_i(q_i, q_j, x_i, x_j)/\delta q_i$:

(Step 3) Due to symmetry, $q_i = q_j$. Use this to find $BR_i = q_i(x_i, x_j)$ and the NE for the 2nd stage.

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Marginal cost of production: $c_{q1} = c_{q2} = c_q = 1 - x_1 - x_2$

3 Cost of research for firm i: $c_{xi} = x_i^2$

4 $Payoff_i(q_i, q_j, x_i, x_j)$: $\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

5 FOC_{q_i}: $1 - 2q_i - q_j + x_i + x_j = 0$

PS6, Ex. 5.a: Cournot, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step a) Write up the payoff function, taking research as given.

(Step b) Write up the FOC and find the best response function for q_i.

(Step 3) Due to symmetry, $q_i = q_j$. Use this to find $BR_i = q_i(x_i, x_j)$ and the NE for the 2nd stage:

$$\begin{aligned} 1 - 2q_i - q_i + x_i + x_j = 0 &\Rightarrow \\ 1 + x_i + x_j = 3q_i &\Rightarrow \\ q_i &= \frac{1 + x_i + x_j}{3} \end{aligned}$$

NE: $(q_1, q_2) = \left(\frac{1+x_1+x_2}{3}, \frac{1+x_1+x_2}{3}\right)$

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Marginal cost of production: $c_{q1} = c_{q2} = c_q = 1 - x_1 - x_2$

3 Cost of research for firm i: $c_{xi} = x_i^2$

4 $Payoff_i(q_i, q_j, x_i, x_j)$: $\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

5 FOC_{q_i}: $1 - 2q_i - q_j + x_i + x_j = 0$

6 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

PS6, Ex. 5.b: Cournot, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm *i* chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

Information so far:

1 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

2 $Payoff_i(q_i, q_j, x_i, x_j)$:
$\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

PS6, Ex. 5.b: Cournot, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step a) Write up the payoff function as a function of research: $\pi_i(x_i, x_j)$.

Information so far:

1 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

2 $Payoff_i(q_i, q_j, x_i, x_j)$:
$\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

PS6, Ex. 5.b: Cournot, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step a) Write up the payoff function as a function of research: $\pi_i(x_i, x_j)$:

$$\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2 \Rightarrow$$

$$= \left(1 + x_i + x_j - 2\frac{1 + x_i + x_j}{3}\right)$$

$$\frac{1 + x_i + x_j}{3} - x_i^2 \Rightarrow$$

$$= \left(\frac{1 + x_i + x_j}{3}\right)\frac{1 + x_i + x_j}{3} - x_i^2 \Rightarrow$$

$$= \frac{(1 + x_i + x_j)^2}{9} - x_i^2$$

Information so far:

1. $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$
2. $Payoff_i(q_i, q_j, x_i, x_j)$:
   $\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

PS6, Ex. 5.b: Cournot, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step a) Write up the payoff function as a function of research: $\pi_i(x_i, x_j)$:

(Step b) Write up the FOC_{x_i} : $\delta\pi_i(x_i, x_j)/\delta x_i$.

Information so far:

1 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

2 $Payoff_i(q_i, q_j, x_i, x_j)$:
$\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

3 $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

PS6, Ex. 5.b: Cournot, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step a) Write up the payoff function as a function of research: $\pi_i(x_i, x_j)$.

(Step b) Write up the FOC_{x_i} :

$$\frac{\delta \pi_i(x_i, x_j)}{\delta x_i} = \frac{2}{9}(1 + x_i + x_j) - 2x_i = 0$$

Information so far:

1. $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$
2. $Payoff_i(q_i, q_j, x_i, x_j)$:
   $\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$
3. $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

PS6, Ex. 5.b: Cournot, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step a) Write up the payoff function as a function of research: $\pi_i(x_i, x_j)$.

(Step b) Write up the FOC_{x_i} :

(Step 3) Use symmetry to find the SPNE by setting $x_i = x_j = x_i^*$, isolating x_i^*, and calculating $q_i(x_i^*, x_j^*)$.

Information so far:

1 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

2 $Payoff_i(q_i, q_j, x_i, x_j)$:
$\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

3 $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

4 FOC_{x_i}: $\frac{2}{9}(1 + x_i + x_j) - 2x_i = 0$

PS6, Ex. 5.b: Cournot, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step a) Write up the payoff function as a function of research: $\pi_i(x_i, x_j)$.

(Step b) Write up the FOC_{x_i} : $\delta\pi_i(x_i, x_j)/\delta x_i$.

(Step 3) Use symmetry to find the SPNE by setting $x_i = x_j = x_i^*$, isolating x_i^*, and calculating $q_i(x_i^*, x_j^*)$:

$$\frac{2}{9}(1 + 2x_i^*) - 2x_i^* = 0 \Rightarrow$$

$$\frac{2}{9} = \frac{14}{9}x_i^* \Rightarrow$$

$$x_i^* = \frac{1}{7}$$

$$q_i^* = q_i(x_i^*, x_j^*) = \frac{1 + \frac{1}{7} + \frac{1}{7}}{3} = \frac{1}{3} \cdot \frac{9}{7} = \frac{3}{7}$$

SPNE: $(x_1^*, x_2^*, q_1^*, q_2^*) = \left(\frac{1}{7}, \frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right)$

Information so far:

1. $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$
2. $Payoff_i(q_i, q_j, x_i, x_j)$: $\pi_i = (1 + x_i + x_j - q_i - q_j)q_i - x_i^2$
3. $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$
4. FOC_{x_i}: $\frac{2}{9}(1 + x_i + x_j) - 2x_i = 0$

PS6, Ex. 5.c: Cournot, colluding to everyone's benefit?

(c) Assume now that the firms collude in the first stage. That is, they choose x_1 and x_2 to maximize their joint profit while taking into account that q_1 and q_2 will be chosen simultaneously and independently in stage two. Find the resulting levels of research and output: $x_1^{**}, x_2^{**}, q_1(x_1^{**}, x_2^{**})$ and $q_2(x_1^{**}, x_2^{**})$.

Information so far:

1 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

2 $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

PS6, Ex. 5.c: Cournot, colluding to everyone's benefit?

(c) Assume now that the firms collude in the first stage. That is, they choose x_1 and x_2 to maximize their joint profit while taking into account that q_1 and q_2 will be chosen simultaneously and independently in stage two. Find the resulting levels of research and output: $x_1^{**}, x_2^{**}, q_1(x_1^{**}, x_2^{**})$ and $q_2(x_1^{**}, x_2^{**})$.

(Step a) Under collusion, write up the total payoff function $\Pi(x_i, x_j)$.

Information so far:

1. $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$
2. $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

PS6, Ex. 5.c: Cournot, colluding to everyone's benefit?

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(Step a) Under collusion, write up the total payoff function $\Pi(x_i, x_j)$:

$$\Pi(x_i, x_j) = \pi_i + \pi_j \Rightarrow$$

$$= 2\frac{(1 + x_i + x_j)^2}{9} - x_i^2 - x_j^2$$

Information so far:

1 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

2 $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

PS6, Ex. 5.c: Cournot, colluding to everyone's benefit?

(c) Assume now that the firms collude in the first stage. That is, they choose x_1 and x_2 to maximize their joint profit while taking into account that q_1 and q_2 will be chosen simultaneously and independently in stage two. Find the resulting levels of research and output: $x_1^{**}, x_2^{**}, q_1(x_1^{**}, x_2^{**})$ and $q_2(x_1^{**}, x_2^{**})$.

(Step a) Under collusion, write up the total payoff function $\Pi(x_i, x_j)$:

(Step b) Write up the $FOC_{x_i} : \delta\Pi(x_i, x_j)/\delta x_i$.

Information so far:

1 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

2 $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

3 $Payoff_{total}(x_i, x_j)$:
$\Pi = 2\frac{(1+x_i+x_j)^2}{9} - x_i^2 - x_j^2$

PS6, Ex. 5.c: Cournot, colluding to everyone's benefit?

(c) Assume now that the firms collude in the first stage. That is, they choose x_1 and x_2 to maximize their joint profit while taking into account that q_1 and q_2 will be chosen simultaneously and independently in stage two. Find the resulting levels of research and output: $x_1^{**}, x_2^{**}, q_1(x_1^{**}, x_2^{**})$ and $q_2(x_1^{**}, x_2^{**})$.

(Step a) Under collusion, write up the total payoff function $\Pi(x_i, x_j)$.

(Step b) Write up the $FOC_{x_i} : \delta\Pi(x_i, x_j)/\delta x_i$.

$$\frac{\delta\Pi(x_i, x_j)}{\delta x_i} = 4\frac{(1 + x_i + x_j)^2}{9} - 2x_i = 0$$

Information so far:

1. $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$
2. $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$
3. $Payoff_{total}(x_i, x_j)$:
   $\Pi = 2\frac{(1+x_i+x_j)^2}{9} - x_i^2 - x_j^2$

PS6, Ex. 5.c: Cournot, colluding to everyone's benefit?

(c) Assume now that the firms collude in the first stage. That is, they choose x_1 and x_2 to maximize their joint profit while taking into account that q_1 and q_2 will be chosen simultaneously and independently in stage two. Find the resulting levels of research and output: $x_1^{**}, x_2^{**}, q_1(x_1^{**}, x_2^{**})$ and $q_2(x_1^{**}, x_2^{**})$.

(Step a) Under collusion, write up the total payoff function $\Pi(x_i, x_j)$.

(Step b) Write up the $FOC_{x_i} : \delta\Pi(x_i, x_j)/\delta x_i$.

(Step 3) Taking advantage of symmetry, find the outcome by isolating x_i^{**} and calculate q_i^{**} using $BR_i(x_i^{**}, x_j^{**})$.

Information so far:

1 $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$

2 $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

3 $Payoff_{total}(x_i, x_j)$:
$\Pi = 2\frac{(1+x_i+x_j)^2}{9} - x_i^2 - x_j^2$

4 $FOC_{x_i} : 4\frac{(1+x_i+x_j)^2}{9} - 2x_i = 0$

PS6, Ex. 5.c: Cournot, colluding to everyone's benefit?

(c) Assume now that the firms collude in the first stage. That is, they choose x_1 and x_2 to maximize their joint profit while taking into account that q_1 and q_2 will be chosen simultaneously and independently in stage two. Find the resulting levels of research and output: $x_1^{**}, x_2^{**}, q_1(x_1^{**}, x_2^{**})$ and $q_2(x_1^{**}, x_2^{**})$.

(Step a) Under collusion, write up the total payoff function $\Pi(x_i, x_j)$.

(Step b) Write up the $FOC_{x_i} : \delta\Pi(x_i, x_j)/\delta x_i$.

(Step 3) Taking advantage of symmetry, find the outcome by isolating x_i^{**} and calculate q_i^{**} using $BR_i(x_i^{**}, x_j^{**})$:

Information so far:

1. $BR_i(x_i, x_j)$: $q_i = \frac{1+x_i+x_j}{3}$
2. $Payoff_i(x_1, x_2)$: $\pi_i = \frac{(1+x_i+x_j)^2}{9} - x_i^2$
3. $Payoff_{total}(x_i, x_j)$:
   $\Pi = 2\frac{(1+x_i+x_j)^2}{9} - x_i^2 - x_j^2$
4. $FOC_{x_i} : 4\frac{(1+x_i+x_j)^2}{9} - 2x_i = 0$

$$4\frac{(1 + x_i^{**} + x_j^{**})^2}{9} - 2x_i^{**} = 0 \Rightarrow$$

$$\frac{4}{9} + \frac{8}{9}x_i^{**} - \frac{18}{9}x_i^{**} = 0 \Rightarrow$$

$$x_i^{**} = \frac{4}{9} \cdot \frac{9}{10} = \frac{4}{10} = \frac{2}{5}$$

$$q_i^{**} = \frac{1 + x_i + x_j}{3} = \frac{1}{3}\left(1 + \frac{2}{5} + \frac{2}{5}\right) = \frac{1}{3} \cdot \frac{9}{5} = \frac{3}{5}$$

Outcome: $(x_1^{**}, x_2^{**}, q_1^{**}, q_2^{**}) = \left(\frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}\right)$

PS6, Ex. 5.d: Cournot, colluding to everyone's benefit?

(d) Based on your findings in (b) and (c), compare the outcomes in terms of consumer welfare [hint: it is enough to look at total output] and firms' profit [hint: no calculations are necessary]. Comment on the source of the difference.

Relevant information:

(b) $(x_T^*, q_T^*) = \left(\frac{2}{7}, \frac{6}{7}\right)$

(c) $(x_T^{**}, q_T^{**}) = \left(\frac{4}{5}, \frac{6}{5}\right)$

P: $2 - q_1 - q_2$

π_i : $(1 + x_i + x_j - q_i - q_j)q_i - x_i^2$

PS6, Ex. 5.d: Cournot, colluding to everyone's benefit?

(d) Based on your findings in (b) and (c), compare the outcomes in terms of consumer welfare [hint: it is enough to look at total output] and firms' profit [hint: no calculations are necessary]. Comment on the source of the difference.

- Since the quantity in (c) is higher, this also means that the price P is lower. Higher quantity and lower price means there is a higher consumer welfare.
- The growth in x_T from collusion is much higher than the growth in q_T from collusion, i.e. the profit in (c) is higher.
- The difference comes from the fact that the collusion in the first stage leads to more research which drives down the marginal cost of production. The benefit of this is distributed amongst companies and consumers.

Relevant information:

(b) $(x^*_{total}, q^*_{total}) = \left(\frac{2}{7}, \frac{6}{7}\right)$

(c) $(x^{**}_{total}, q^{**}_{total}) = \left(\frac{4}{5}, \frac{6}{5}\right)$

P: $2 - q_1 - q_2$

π_i : $(1 + x_i + x_j - q_i - q_j)q_i - x_i^2$