Microeconomics III: Problem Set 7[a]

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[a] Slides created for exercise class reservation for possible errors.

Outline

PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

PS7, Ex. 2 (A): A single stage game NE (finitely repeated game)

PS7, Ex. 3: Trigger strategy (infinitely repeated game)

PS7, Ex. 4: Credible punishment (twice-repeated game)

PS7, Ex. 5: Tit-for-tat strategy (infinitely repeated game)

PS7, Ex. 6: Bertrand duopoly (infinitely repeated game)

PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

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Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

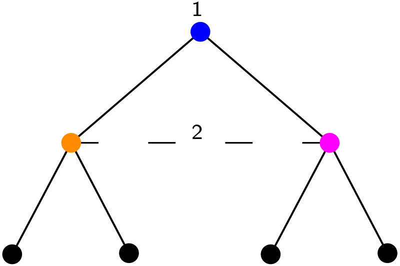
PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.

Example of violation of condition 1:



PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

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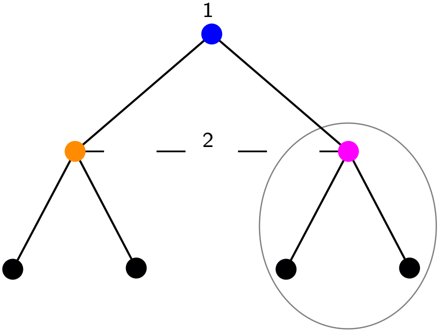


The purple decision node to the right is not a singleton information set (nor is the orange decision node to the left).

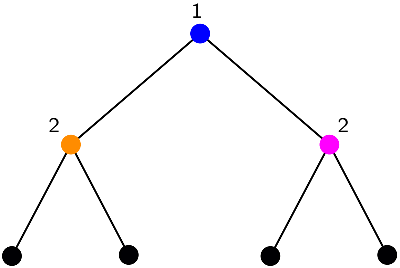
PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. **It includes all following decision and terminal nodes following *n* in the game tree**, but no nodes that do not follow *n*.

Example of violation of the first part of condition 2:



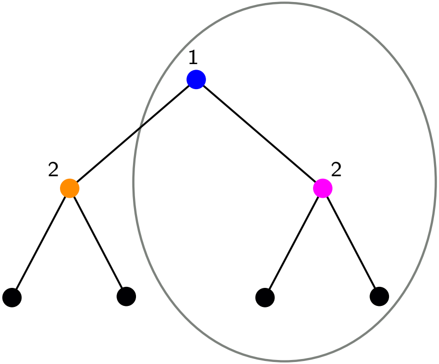
PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. **It includes all following decision and terminal nodes following *n* in the game tree**, but no nodes that do not follow *n*.

Example of violation of the first part of condition 2:



For a subgame containing the blue decision node *n*, all following decision nodes must be included.

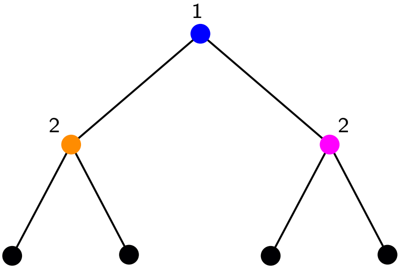
PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. It includes all following decision and terminal nodes following *n* in the game tree, **but no nodes that do not follow *n*.**

Example of violation of the second part of condition 2:



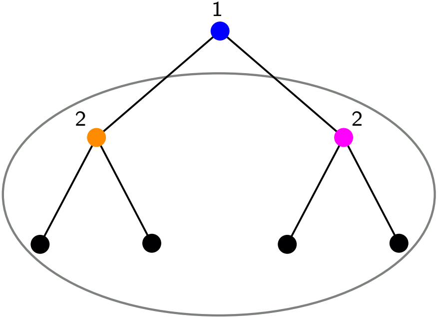
PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. It includes all following decision and terminal nodes following *n* in the game tree, **but no nodes that do not follow *n*.**

Example of violation of the second part of condition 2:



Regardless of whether the orange or the purple node is chosen as the first decision node *n*, the other decision node does not follow *n*, and therefore cannot be part of the subgame.

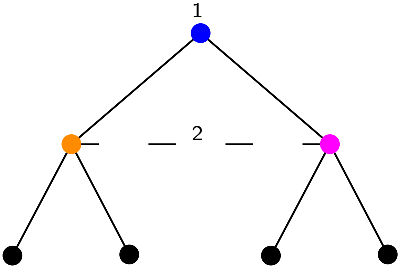
PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node n that is a singleton information set.
2. It includes all following decision and terminal nodes following n in the game tree, but no nodes that do not follow n.
3. It does not "cut" any information set: if a decision node n' follows n in the game tree, then all other nodes in the information set including n' must also follow n (and so be included in the subgame).

Example of violation of condition 3:



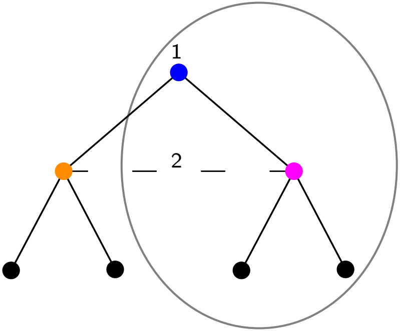
PS7, Ex. 1 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. It includes all following decision and terminal nodes following *n* in the game tree, but no nodes that do not follow *n*.
3. It does not "cut" any information set: if a decision node n' follows *n* in the game tree, then all other nodes in the information set including n' must also follow *n* (and so be included in the subgame).

Example of violation of condition 3:



The orange decision node to the left is part of the same information set as the purple node to the right, so it must be included in the same subgame.

PS7, Ex. 2 (A): A single stage game NE (finitely repeated game)

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Let G be the following game:

		Player 2	
		C	D
Player 1	A	27, -3	0, 0
	B	6, 6	-2, 7

Consider the repeated game G(T), where G is repeated T times and the outcomes of each round are observed by both players before the next round.

(a) If T = 2, is there a Subgame Perfect Nash Equilibrium such that (B,C) is played during the 1st round?

(b) What if T = 42?

PS7, Ex. 2.a (A): A single stage game NE (finitely repeated game)

(a) If T = 2, is there a Subgame Perfect Nash Equilibrium such that (B,C) is played during the 1st round?

		Player 2	
		C	D
Player 1	A	27, -3	0, 0
	B	6, 6	-2, 7

No. Since there is only one NE (A,D) which is not (B,C), that NE will be played in both games.

Explanation:
In the last round, a NE from the stage game must be played. In this case there is only one NE, which is (A,D). Knowing that (A,D) will be played no matter what in the 2nd round, no player has an incentive to cooperate in the 1st turn. Player A will play his dominant strategy A and player B will play her dominant strategy D.

PS7, Ex. 2.b (A): A single stage game NE (finitely repeated game)

(b) If T = 42, is there a Subgame Perfect Nash Equilibrium such that (B,C) is played during the 1st round?

		Player 2	
		C	D
Player 1	A	27, -3	0, 0
	B	6, 6	-2, 7

PS7, Ex. 2.b (A): A single stage game NE (finitely repeated game)

(b) If T = 42, is there a Subgame Perfect Nash Equilibrium such that (B,C) is played during the 1st round?

Player 2

Player 1	C	D
A	27, -3	0, 0
B	6, 6	-2, 7

No. Since there is only one NE (A,D) which is not (B,C), that NE will be played in every turn of any finite game G(T).

Explanation:
In the last round, an NE from the stage game must be played. In this case there is only one NE, which is (A,D). Knowing that (A,D) will be played no matter what in the last round, no player has an incentive to cooperate in the round before that. This keeps applying until the players reach the 1st stage of the game. Thus, the NE (A,D) will be played in every turn of any finite game G(T).

PS7, Ex. 3: Trigger strategy (infinitely repeated game)

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Consider the situation of two flatmates. They both prefer having a clean kitchen, but cleaning is a tedious task, so that it is individually rational not to clean regardless of what the other does. This results in the following game G:

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	DCl	5, 0	1, 1

Now consider the situation where the two flatmates have to decide every day whether to clean or not, i.e. consider the infinitely repeated game $G(\infty, \delta)$

(a) Define trigger strategies such that the outcome of all stages will be (Clean,Clean).

(b) Find the lowest value of δ such that the trigger strategies from (b) constitute a SPNE in $G(\infty, \delta)$. Recall: you have to check for deviations both on and off the equilibrium path.

PS7, Ex. 3.a: Trigger strategy (infinitely repeated game)

Consider the infinitely repeated game $G(\infty, \delta)$ with the stage game:

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	DCl	5, 0	1, 1

(a) Define trigger strategies such that the outcome of all stages will be (Clean,Clean).

A trigger strategy is defined as the player will play the same option in every game (the carrot), unless the opponent does something (the trigger), then he will play something else (the stick).

1 Define the carrot, the trigger and the stick.

PS7, Ex. 3.a: Trigger strategy (infinitely repeated game)

Consider the infinitely repeated game $G(\infty, \delta)$ with the stage game:

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	DCl	5, 0	1, 1

(a) Define trigger strategies such that the outcome of all stages will be (Clean,Clean).

A trigger strategy is defined as the player will play the same option in every game (the carrot), unless the opponent does something (the trigger), then he will play something else (the stick).

1 Define the carrot, the trigger and the stick.

2 Write up the trigger strategy

1. Carrot: Playing Clean
2. Trigger: if the other player doesn't play Clean
3. Stick: Playing Don't Clean

PS7, Ex. 3.a: Trigger strategy (infinitely repeated game)

Consider the infinitely repeated game $G(\infty, \delta)$ with the stage game:

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	DCl	5, 0	1, 1

(a) Define trigger strategies such that the outcome of all stages will be (Clean,Clean).

A trigger strategy is defined as the player will play the same option in every game (the carrot), unless the opponent does something (the trigger), then he will play something else (the stick).

1 Define the carrot, the trigger and the stick.

2 Write up the trigger strategy

1. Carrot: Playing Clean
2. Trigger: if the other player doesn't play Clean
3. Stick: Playing Don't Clean
4. Trigger strategy: In the 1st turn, play Clean. In every subsequent turn, if outcome from every previous turn was (Clean,Clean), play Clean, otherwise play Don't Clean.

PS7, Ex. 3.b: Trigger strategy (infinitely repeated game)

		Player 2	
		CI	DCI
Player 1	CI	4, 4	0, 6
	DCI	5, 0	1, 1

(b) Find the lowest value of δ such that the trigger strategies from (b) constitute a SPNE in $G(\infty, \delta)$. Recall: you have to check for deviations both on and off the equilibrium path.

PS7, Ex. 3.b: Trigger strategy (infinitely repeated game)

(b) Find the lowest value of δ such that the trigger strategies from (b) constitute a SPNE in $G(\infty, \delta)$. Recall: you have to check for deviations both on and off the equilibrium path.

(Step a) *On the equilibrium path:* Define the payoff for staying with the trigger strategy, and for deviating, then write up the inequality and isolate δ to find for what values of δ Player 2 wouldn't deviate, you only need to check P2 as P2 has the highest incentive to deviate.

1. $U_2(CI, CI) = 4$
2. $U_2(CI, DCI) = 6$
3. $U_2(DCI, DCI) = 1$
4. Algebra of infinite sequences:
   $\sum_{t=1}^{\infty} a \cdot \delta^{t-1} = \frac{a}{1-\delta}$
   $\sum_{t=2}^{\infty} a \cdot \delta^{t-1} = \frac{a\delta}{1-\delta}$

PS7, Ex. 3.b: Trigger strategy (infinitely repeated game)

(b) Find the lowest value of δ such that the trigger strategies from (b) constitute a SPNE in $G(\infty, \delta)$. Recall: you have to check for deviations both on and off the equilibrium path.

(Step a) *On the equilibrium path:* Define the payoff for staying with the trigger strategy, and for deviating, then write up the inequality and isolate δ to find for what values of δ Player 2 wouldn't deviate, you only need to check P2 as P2 has the highest incentive to deviate.

(Step b) *Off the equilibrium path:*

1. $U_2(CI, CI) = 4$
2. $U_2(CI, DCI) = 6$
3. $U_2(DCI, DCI) = 1$
4. Algebra of infinite sequences: $\sum_{t=1}^{\infty} a \cdot \delta^{t-1} = \frac{a}{1-\delta}$ $\sum_{t=2}^{\infty} a \cdot \delta^{t-1} = \frac{a\delta}{1-\delta}$
5. On the equilibrium path:

$$4 + 4\delta + 4\delta^2 + ... \geq 6 + 1\delta + 1\delta^2 + ... \Rightarrow$$

$$4\delta^0 + 4\delta + 4\delta^2 + ... \geq 6 + 1\delta + 1\delta^2 + ... \Rightarrow$$

$$\sum_{t=1}^{\infty} 4 \cdot \delta^{t-1} \geq 6 + \sum_{\boldsymbol{t=2}}^{\infty} 1 \cdot \delta^{t-2} \Rightarrow$$

$$\frac{4}{1-\delta} \geq 6 + \frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{2}{5}$$

PS7, Ex. 3.b: Trigger strategy (infinitely repeated game)

(b) Find the lowest value of δ such that the trigger strategies from (b) constitute a SPNE in $G(\infty, \delta)$. Recall: you have to check for deviations both on and off the equilibrium path.

(Step a) *On the equilibrium path:* Define the payoff for staying with the trigger strategy, and for deviating, then write up the inequality and isolate δ to find for what values of δ Player 2 wouldn't deviate, you only need to check P2 as P2 has the highest incentive to deviate.

(Step b) *Off the equilibrium path:* Check if the trigger strategy is credible if a player deviated from the equilibrium path by playing "don't clean" in the previous round.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$
4. Algebra of infinite sequences: $\sum_{t=1}^{\infty} a \cdot \delta^{t-1} = \frac{a}{1-\delta}$ $\sum_{t=2}^{\infty} a \cdot \delta^{t-1} = \frac{a\delta}{1-\delta}$
5. On the equilibrium path:

$$4 + 4\delta + 4\delta^2 + ... \geq 6 + 1\delta + 1\delta^2 + ... \Rightarrow$$

$$4\delta^0 + 4\delta + 4\delta^2 + ... \geq 6 + 1\delta + 1\delta^2 + ... \Rightarrow$$

$$\sum_{t=1}^{\infty} 4 \cdot \delta^{t-1} \geq 6 + \sum_{\boldsymbol{t=2}}^{\infty} 1 \cdot \delta^{t-2} \Rightarrow$$

$$\frac{4}{1-\delta} \geq 6 + \frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{2}{5}$$

PS7, Ex. 3.b: Trigger strategy (infinitely repeated game)

(b) Find the lowest value of δ such that the trigger strategies from (b) constitute a SPNE in $G(\infty, \delta)$. Recall: you have to check for deviations both on and off the equilibrium path.

(Step a) *On the equilibrium path:* Define the payoff for staying with the trigger strategy, and for deviating, then write up the inequality and isolate δ to find for what values of δ Player 2 wouldn't deviate, you only need to check P2 as P2 has the highest incentive to deviate.

(Step b) *Off the equilibrium path:* Check if the trigger strategy is credible if a player deviated from the equilibrium path by playing "don't clean" in the previous round.

The best response to "don't clean" is to also play "don't clean". As (DCl,DCl) is the stage game NE, this is a credible punishment as there is no incentive to deviate from this eternal punishment.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$
4. Algebra of infinite sequences:
   $\sum_{t=1}^{\infty} a \cdot \delta^{t-1} = \frac{a}{1-\delta}$
   $\sum_{t=2}^{\infty} a \cdot \delta^{t-1} = \frac{a\delta}{1-\delta}$
5. On the equilibrium path:
$$4 + 4\delta + 4\delta^2 + ... \geq 6 + 1\delta + 1\delta^2 + ... \Rightarrow$$
$$4\delta^0 + 4\delta + 4\delta^2 + ... \geq 6 + 1\delta + 1\delta^2 + ... \Rightarrow$$
$$\sum_{t=1}^{\infty} 4 \cdot \delta^{t-1} \geq 6 + \sum_{\boldsymbol{t=2}}^{\infty} 1 \cdot \delta^{t-2} \Rightarrow$$
$$\frac{4}{1-\delta} \geq 6 + \frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{2}{5}$$
6. Neither player will deviate for $\delta \geq \frac{2}{5}$

PS7, Ex. 4: Credible punishment (twice-repeated game)

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Consider the two times repeated game where the stage game is:

Player 2

		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (B,Y). Make sure to write down the full equilibrium.

(b) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (C,Z). Make sure to write down the full equilibrium.

(c) Can you find a subgame perfect Nash equilibrium such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

PS7, Ex. 4.a: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (B,Y). Make sure to write down the full equilibrium.

(Step a) Find the NE in the stage game.

PS7, Ex. 4.a: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

Player 2

Player 1

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (B,Y). Make sure to write down the full equilibrium.

(Step a) Find the NE in the stage game.

Information so far:

1. Stage game NE: $\{(B, Y), (C, Z)\}$

PS7, Ex. 4.a: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

Player 2

		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (B,Y). Make sure to write down the full equilibrium.

(Step a) Find the NE in the stage game.

(Step b) Knowing the NE, is a SPNE possible with (B,Y) in the 1st stage?

Information so far:

1. Stage game NE: $\{(B, Y), (C, Z)\}$

PS7, Ex. 4.a: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

Player 2

Player 1	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (B,Y). Make sure to write down the full equilibrium.

(Step a) Find the NE in the stage game.

(Step b) Knowing the NE, is a SPNE possible with (B,Y) in the 1st stage?

As (B,Y) is a NE it is a possible outcome in any stage. We can choose the strategies such that (B,Y) will be the outcome of the 1st stage, and then either of the NE can be the outcome of the 2nd stage.

Information so far:

1. Stage game NE: $\{(B, Y), (C, Z)\}$

PS7, Ex. 4.a: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (B,Y). Make sure to write down the full equilibrium.

(Step a) Find the NE in the stage game.

(Step b) Knowing the NE, is a SPNE possible with (B,Y) in the 1st stage?

As (B,Y) is a NE it is a possible outcome in any stage. We can choose the strategies such that (B,Y) will be the outcome of the 1st stage, and then either of the NE can be the outcome of the 2nd stage.

(Step c) Write up a possible SPNE.

Information so far:

1. Stage game NE: $\{(B,Y),(C,Z)\}$

PS7, Ex. 4.a: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

Player 2

		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (B,Y). Make sure to write down the full equilibrium.

(Step a) Find the NE in the stage game.

(Step b) Knowing the NE, is a SPNE possible with (B,Y) in the 1st stage?

As (B,Y) is a NE it is a possible outcome in any stage. We can choose the strategies such that (B,Y) will be the outcome of the 1st stage, and then either NE can be the outcome of the 2nd stage.

(Step c) Write up a possible SPNE.

Keep in mind that you need to write up a 2nd stage strategy for each of the possible outcomes of the 1st stage (3·3 matrix, so 9 possible outcomes).

Information so far:

1. Stage game NE: $\{(B, Y), (C, Z)\}$

PS7, Ex. 4.a: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (B,Y). Make sure to write down the full equilibrium.

(Step a) Find the NE in the stage game.

(Step b) Knowing the NE, is a SPNE possible with (B,Y) in the 1st stage?

As (B,Y) is a NE it is a possible outcome in any stage. We can choose the strategies such that (B,Y) will be the outcome of the 1st stage, and then either NE can be the outcome of the 2nd stage.

(Step c) Write up a possible SPNE.

Keep in mind that you need to write up a 2nd stage strategy for each of the possible outcomes of the 1st stage (3·3 matrix, so 9 possible outcomes).

Information so far:

1. Stage game NE: $\{(B, Y), (C, Z)\}$
2. Write up one of 2^9 possible SPNE:

$$\left\{ \begin{array}{c} (BBBBBBBBBB, YYYYYYYYYY) \\ (BCBBBBBBBB, YZYYYYYYYY) \\ \vdots \\ (BCCCCCCCCC, YZZZZZZZZZ) \end{array} \right\}$$

PS7, Ex. 4.b: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(b) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (C,Z). Make sure to write down the full equilibrium.

Information so far:

1. Stage game NE: $\{(B, Y), (C, Z)\}$

PS7, Ex. 4.b: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(b) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (C,Z). Make sure to write down the full equilibrium.

(Step a) Knowing the NE, is a SPNE possible with (C,Z) in the 1st stage?

Information so far:

1. Stage game NE: $\{(B,Y),(C,Z)\}$

PS7, Ex. 4.b: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

Player 1 \ Player 2	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(b) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (C,Z). Make sure to write down the full equilibrium.

(Step a) Knowing the NE, is a SPNE possible with (C,Z) in the 1st stage?

Yes, similarly to question (a), any NE can be played in either round.

Information so far:

1. Stage game NE: $\{(B, Y), (C, Z)\}$

PS7, Ex. 4.b: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(b) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (C,Z). Make sure to write down the full equilibrium.

(Step a) Knowing the NE, is a SPNE possible with (C,Z) in the 1st stage?

Yes, similarly to question (a), any NE can be played in either round.

(Step b) Write up a possible SPNE.

Information so far:

1. Stage game NE: $\{(B, Y), (C, Z)\}$

PS7, Ex. 4.b: Credible punishment (twice-repeated game)

Consider the two times repeated game where the stage game is:

Player 2

		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(b) Find a subgame perfect Nash equilibrium such that the outcome of the 1st stage is (C,Z). Make sure to write down the full equilibrium.

(Step a) Knowing the NE, is a SPNE possible with (C,Z) in the 1st stage?

Yes, similarly to question (a), any NE can be played in either round.

(Step b) Write up a possible SPNE.

Information so far:

1. Stage game NE: $\{(B,Y),(C,Z)\}$
2. Write up one of 2^9 possible SPNE:

$$\left\{ \begin{array}{c} (CBBBBBBBBB, ZYYYYYYYYY) \\ (CCBBBBBBBB, ZZYYYYYYYY) \\ \vdots \\ (CCCCCCCCCC, ZZZZZZZZZZ) \end{array} \right\}$$

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

Player 2

		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

a. Total payoffs are (10,11) for:

t=1: (A,X) (not a stage game NE)
t=2: (C,Z) (a stage game NE)

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

a. Total payoffs are (10,11) for:

t=1: (A,X) (not a stage game NE)

t=2: (C,Z) (a stage game NE)

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

a. Total payoffs are (10,11) for:

t=1: (A,X) (not a stage game NE)

t=2: (C,Z) (a stage game NE)

b. Punishment Strategy *PS*:

t=1: Play (A,X) .

t=2: If (A,X) was played in t=1, play (C,Z). Otherwise, play (B,Y).

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

(Step c) Check if either player is better off from his optimal deviation strategy *OD* than from the punishment strategy *PS*.

a. Total payoffs are (10,11) for:

t=1: (A,X) (not a stage game NE)
t=2: (C,Z) (a stage game NE)

b. Punishment Strategy *PS*:

t=1: Play (A,X) .
t=2: If (A,X) was played in t=1, play (C,Z). Otherwise, play (B,Y).

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
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(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

(Step c) Check if either player is better off from his optimal deviation strategy *OD* than from the punishment strategy *PS*.

a. Total payoffs are (10,11) for:
  - t=1: (A,X) (not a stage game NE)
  - t=2: (C,Z) (a stage game NE)

b. Punishment Strategy *PS*:
  - t=1: Play (A,X) .
  - t=2: If (A,X) was played in t=1, play (C,Z). Otherwise, play (B,Y).

c. P1: Check *PS* is better than his optimal deviation $OD_1 = (B, B)$:

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

(Step c) Check if either player is better off from his optimal deviation strategy *OD* than from the punishment strategy *PS*.

a. Total payoffs are (10,11) for:

t=1: (A,X) (not a stage game NE)
t=2: (C,Z) (a stage game NE)

b. Punishment Strategy *PS*:

t=1: Play (A,X) .
t=2: If (A,X) was played in t=1, play (C,Z). Otherwise, play (B,Y).

c. P1: Check *PS* is better than his optimal deviation $OD_1 = (B, B)$:

$$U_1(PS, PS) \geq U_1(OD_1, PS) \Leftrightarrow 6 + 4 \geq 7 + 2$$

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

(Step c) Check if either player is better off from his optimal deviation strategy *OD* than from the punishment strategy *PS*.

a. Total payoffs are (10,11) for:

t=1: (A,X) (not a stage game NE)

t=2: (C,Z) (a stage game NE)

b. Punishment Strategy *PS*:

t=1: Play (A,X) .

t=2: If (A,X) was played in t=1, play (C,Z). Otherwise, play (B,Y).

c. P1: Check *PS* is better than his optimal deviation $OD_1 = (B, B)$:

$$U_1(PS, PS) \geq U_1(OD_1, PS) \Leftrightarrow 6 + 4 \geq 7 + 2$$

c. P2: Check *PS* vs. $OD_2 = (Y, Y)$:

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

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a. Total payoffs are (10,11) for:

t=1: (A,X) (not a stage game NE)

t=2: (C,Z) (a stage game NE)

b. Punishment Strategy *PS*:

t=1: Play (A,X) .

t=2: If (A,X) was played in t=1, play (C,Z). Otherwise, play (B,Y).

c. P1: Check *PS* is better than his optimal deviation $OD_1 = (B, B)$:

$$U_1(PS, PS) \geq U_1(OD_1, PS) \Leftrightarrow 6 + 4 \geq 7 + 2$$

c. P2: Check *PS* vs. $OD_2 = (Y, Y)$:

$$U_2(PS, PS) \geq U_2(PS, OD_2) \Leftrightarrow 6 + 5 \geq 8 + 2$$

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: δ=1).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

(Step c) Check if either player is better off from his optimal deviation strategy *OD* than from the punishment strategy *PS*.

As *PS* is a best response to *PS* for both players, *(PS,PS)* is a SPNE.

(Step d) Write up the full SPNE.

a. Total payoffs are (10,11) for:

t=1: (A,X) (not a stage game NE)

t=2: (C,Z) (a stage game NE)

b. Punishment Strategy *PS*:

t=1: Play (A,X) .

t=2: If (A,X) was played in t=1, play (C,Z). Otherwise, play (B,Y).

c. P1: Check *PS* is better than his optimal deviation $OD_1 = (B, B)$:

$$U_1(PS, PS) \geq U_1(OD_1, PS) \Leftrightarrow 6 + 4 \geq 7 + 2$$

c. P2: Check *PS* vs. $OD_2 = (Y, Y)$:

$$U_2(PS, PS) \geq U_2(PS, OD_2) \Leftrightarrow 6 + 5 \geq 8 + 2$$

PS7, Ex. 4.c: Credible punishment (twice-repeated game)

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

(Step a) Find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be a NE (and perfect patience: $\delta = 1$).

(Step b) Now, look for a punishment strategy *PS* which if followed will lead to this combination.

(Step c) Check if either player is better off from his optimal deviation strategy *OD* than from the punishment strategy *PS*.

As *PS* is a best response to *PS* for both players, (PS, PS) is a SPNE.

(Step d) Write up the full SPNE.

a. Total payoffs are (10,11) for:

$t=1$: (A,X) (not a stage game NE)
$t=2$: (C,Z) (a stage game NE)

b. Punishment Strategy *PS*:

$t=1$: Play (A,X) .
$t=2$: If (A,X) was played in $t=1$, play (C,Z). Otherwise, play (B,Y).

c. P1: Check *PS* is better than his optimal deviation $OD_1 = (B, B)$:

$$U_1(PS, PS) \geq U_1(OD_1, PS) \Leftrightarrow 6 + 4 \geq 7 + 2$$

c. P2: Check *PS* vs. $OD_2 = (Y, Y)$:

$$U_2(PS, PS) \geq U_2(PS, OD_2) \Leftrightarrow 6 + 5 \geq 8 + 2$$

d. SPNE:

$\{(ACBBBBBBBB, XZYYYYYYYY)\}$

PS7, Ex. 5: Tit-for-tat strategy (infinitely repeated game)

PS7, Ex. 5: Tit-for-tat strategy (infinitely repeated game)

Consider again the the infinitely repeated game $G(\infty, \delta)$ with the stage game:

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

(a) Define a tit-for-tat strategy such that the outcome of all stages will be (Clean, Clean).

(b) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Always play 'Do not clean"

(c) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Start by playing 'Do not clean', then play 'tit-for-tat' forever after that'.

(d) Argue informally that 'tit-for-tat' is a NE for the appropriate values of δ. In particular, think about whether there are other deviations that would be better for the players.

(e) Is tit-for-tat a SPNE?

When we say "against", it doesn't mean that the other player is playing the "against" strategy. It means to compare the two strategies, in this case "on the equilibrium path", so if the other player is playing "tit-for-tat"

PS7, Ex. 5.a: Tit-for-tat strategy (infinitely repeated game)

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

(a) Define a tit-for-tat strategy such that the outcome of all stages will be (Clean, Clean).

A tit-for-tat strategy is defined as a strategy where the player plays the carrot option, if it's the 1^{st} round or the other player played the carrot option in the last round, otherwise the player will play the stick option.

(Step a) Define the carrot and the stick.

PS7, Ex. 5.a: Tit-for-tat strategy (infinitely repeated game)

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

(a) Define a tit-for-tat strategy such that the outcome of all stages will be (Clean, Clean).

A tit-for-tat strategy is defined as a strategy where the player plays the carrot option, if it's the 1$^{\text{st}}$ round or the other player played the carrot option in the last round, otherwise the player will play the stick option.

(Step a) Define the carrot and the stick.

(Step b) Write up the tit-for-tat strategy

1. Carrot: Playing Clean
2. Stick: Playing Don't Clean

PS7, Ex. 5.a: Tit-for-tat strategy (infinitely repeated game)

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

(a) Define a tit-for-tat strategy such that the outcome of all stages will be (Clean, Clean).

A tit-for-tat strategy is defined as a strategy where the player plays the carrot option, if it's the 1st round or the other player played the carrot option in the last round, otherwise the player will play the stick option.

1. Carrot: Playing Clean
2. Stick: Playing Don't Clean

(Step a) Define the carrot and the stick.

(Step b) Write up the tit-for-tat strategy

PS7, Ex. 5.b: Tit-for-tat strategy (infinitely repeated game)

(b) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Always play 'Do not clean''

(Step a) Define the payoff for staying with the tit-for-tat strategy, and for deviating.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$

PS7, Ex. 5.b: Tit-for-tat strategy (infinitely repeated game)

(b) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Always play 'Do not clean''

(Step a) Define the payoff for staying with the tit-for-tat strategy, and for deviating.

(Step b) Write up the inequality and isolate δ to find for what values of δ neither player would deviate.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$
4. On the equilibrium path:

PS7, Ex. 5.b: Tit-for-tat strategy (infinitely repeated game)

(b) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Always play 'Do not clean''

(Step a) Define the payoff for staying with the tit-for-tat strategy, and for deviating.

(Step b) Write up the inequality and isolate δ to find for what values of δ neither player would deviate, you only need to check P2 as P2 has the highest incentive to deviate.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$
4. On the equilibrium path:

PS7, Ex. 5.b: Tit-for-tat strategy (infinitely repeated game)

(b) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Always play 'Do not clean''

(Step a) Define the payoff for staying with the tit-for-tat strategy, and for deviating.

(Step b) Write up the inequality and isolate δ to find for what values of δ player 2 wouldn't deviate, you only need to check P2 as P2 has the highest incentive to deviate.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$
4. On the equilibrium path:

$$4 + 4\delta + 4\delta^2 + ... \geq 6 + 1\delta + 1\delta^2 + ... \Rightarrow$$
$$4\delta^0 + 4\delta + 4\delta^2 + ... \geq 6 + 1\delta + 1\delta^2 + ... \Rightarrow$$
$$\sum_{t=1}^{\infty} 4 \cdot \delta^{t-1} \geq 6 + \sum_{\boldsymbol{t=2}}^{\infty} 1 \cdot \delta^{t-1} \Rightarrow$$
$$\frac{4}{1-\delta} \geq 6 + \frac{\delta}{1-\delta} \Rightarrow$$
$$\delta \geq \frac{2}{5}$$

5. Neither player will deviate for $\delta \geq \frac{2}{5}$

PS7, Ex. 5.c: Tit-for-tat strategy (infinitely repeated game)

(c) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Start by playing 'Do not clean', then play 'tit-for-tat' forever after that'.

(Step a) Define the payoff for staying with the tit-for-tat strategy, and for deviating. Then write up the inequality and isolate δ to find for what values of δ player 2 wouldn't deviate, you only need to check P2 as P2 has the highest incentive to deviate.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$
4. On the equilibrium path:

PS7, Ex. 5.c: Tit-for-tat strategy (infinitely repeated game)

(c) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Start by playing 'Do not clean', then play 'tit-for-tat' forever after that'.

(Step a) Define the payoff for staying with the tit-for-tat strategy, and for deviating. Then write up the inequality and isolate δ to find for what values of δ player 2 wouldn't deviate, you only need to check P2 as P2 has the highest incentive to deviate.

In the case where the P2 deviates, the outcome in round 1 will be (clean,don't clean), in the next round, following his tit-for-tat strategy, P1 will play don't clean. P2 will switch to his tit-for-tat strategy and play clean. The outcome in round 2 will be (Don't clean,clean) and in round 3 the (clean, don't clean), continuing this pattern.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$
4. On the equilibrium path:

$$4 + 4\delta + ... \geq 6 + 0\delta + 6\delta^2 + 0\delta^3 + 6\delta^4 ... \Rightarrow$$

$$4 + 4\delta + ... \geq 6\delta^0 + 6\delta^2 + 6\delta^4 + ... \Rightarrow$$

$$\sum_{t=1}^{\infty} 4 \cdot \delta^{t-1} \geq \sum_{t=1}^{\infty} 6 \cdot \delta^{2(t-1)} \Rightarrow$$

$$\sum_{t=1}^{\infty} 4 \cdot \delta^{t-1} \geq \sum_{t=1}^{\infty} 6 \cdot (\delta^2)^{t-1} \Rightarrow$$

$$\frac{4}{1-\delta} \geq \frac{6}{1-\delta^2} \Rightarrow$$

$$-2\delta^2 + 3\delta - 1 \geq 0$$

PS7, Ex. 5.c: Tit-for-tat strategy (infinitely repeated game)

(c) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Start by playing 'Do not clean', then play 'tit-for-tat' forever after that'.

(Step a) Define the payoff for staying with the tit-for-tat strategy, and for deviating. Then write up the inequality and isolate δ to find for what values of δ player 2 wouldn't deviate, you only need to check P2 as P2 has the highest incentive to deviate.

In the case where the P2 deviates, the outcome in round 1 will be (clean,don't clean), in the next round, following his tit-for-tat strategy, P1 will play don't clean. P2 will switch to his tit-for-tat strategy and play clean. The outcome in round 2 will be (Don't clean,clean) and in round 3 the (clean, don't clean), continuing this pattern.

1. $U_2(Cl, Cl) = 4$
2. $U_2(Cl, DCl) = 6$
3. $U_2(DCl, DCl) = 1$
4. On the equilibrium path:

$$4 + 4\delta + ... \geq 6 + 0\delta + 6\delta^2 + 0\delta^3 + 6\delta^4 ... \Rightarrow$$

$$\sum_{t=1}^{\infty} 4 \cdot \delta^{t-1} \geq \sum_{t=1}^{\infty} 6 \cdot (\delta^2)^{t-1} \Rightarrow$$

$$\frac{4}{1-\delta} \geq \frac{6}{1-\delta^2} \Rightarrow$$

$$-2\delta^2 + 3\delta - 1 \geq 0$$

1. This is a 2$^{\text{nd}}$ degree polynomial which is equal to 0 at $\delta = \frac{1}{2}$ and $\delta = 1$. In between it is positive. I.e. neither player will deviate to the proposed strategy for $\delta \in \left[\frac{1}{2}, 1\right]$.

PS7, Ex. 5.d: Tit-for-tat strategy (infinitely repeated game)

Consider again the the infinitely repeated game $G(\infty, \delta)$ with the stage game:

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

(d) Argue informally that 'tit-for-tat' is a NE for the appropriate values of δ. In particular, think about whether there are other deviations that would be better for the players.

PS7, Ex. 5.d: Tit-for-tat strategy (infinitely repeated game)

Consider again the the infinitely repeated game $G(\infty, \delta)$ with the stage game:

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

(d) Argue informally that 'tit-for-tat' is a NE for the appropriate values of δ. In particular, think about whether there are other deviations that would be better for the players.

For $\delta \geq \frac{1}{2}$ we have shown that tit-for-tat is better than the two deviations. If one of the players were to apply the trigger strategy or "always play clean", the outcome would be the same as for playing tit-for-that, which is (clean,clean) in every round.

Of the two deviations, for $\delta \geq \frac{1}{2}$ the "play don't clean then tit for tat" dominates the "always play don't clean". This is seen by looking at the payoff of the 2nd and 3rd round (1st round is the same). $1 + \frac{1}{2} \cdot 1 \leq 0 + 6 \cdot \frac{1}{2}$ the 2nd and 3rd round is essentially repeated forever, so if the payoff for the 2nd and 3rd round is higher, then the sum of the payoffs are higher.

Could other deviations be better? What is required for a strategy to be part of a NE?

PS7, Ex. 5.d: Tit-for-tat strategy (infinitely repeated game)

Consider again the the infinitely repeated game $G(\infty, \delta)$ with the stage game:

Player 2

		Cl	DCl
Player 1	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

(d) Argue informally that 'tit-for-tat' is a NE for the appropriate values of δ. In particular, think about whether there are other deviations that would be better for the players.

For $\delta \geq \frac{1}{2}$ we have shown that tit-for-tat is better than the two deviations. If one of the players were to apply the trigger strategy or "always play clean", the outcome would be the same as for playing tit-for-that, which is (clean,clean) in every round.

Of the two deviations, for $\delta \geq \frac{1}{2}$ the "play don't clean then tit for tat" dominates the "always play don't clean". This is seen by looking at the payoff of the 2^{nd} and 3^{rd} round (1^{st} round is the same). $1 + \frac{1}{2} \cdot 1 \leq 0 + 6 \cdot \frac{1}{2}$ the 2^{nd} and 3^{rd} round is essentially repeated forever, so if the payoff for the 2^{nd} and 3^{rd} round is higher, then the sum of the payoffs are higher.

The final piece of the puzzle is to realize that all other plausible deviations are combinations of the two deviations we have already examined. Thus, for $\delta \geq \frac{1}{2}$ no deviation can give a strictly higher payoff and 'tit-for-tat' is best-response *on* the equilibrium path which is the requirement for being part of a NE.

PS7, Ex. 5.e: Tit-for-tat strategy (infinitely repeated game)

Player 2

Player 1	Cl	DCl
Cl	4, 4	0, 6
Dcl	5, 0	1, 1

(e) Is tit-for-tat a SPNE?

From the perspective of player 1 (Tit-for-tat, tit-for-tat) would be an SPNE only for the special case $4 + \delta \frac{4}{1-\delta} = 5 + \delta \frac{\delta 5}{1-\delta^2}$ where player 1 is indifferent between playing Cl and Dcl. When the players have different payoffs, they can't both be indifferent at the same time.
Below is the payoff matrix as it looks when both players are playing tit-for-tat. Given that the previous round for player 2 for Dcl (Cl for player 1), this dictates that Dcl, Cl must be an NE in order for tit-for-tat to be an SPNE.
But if the previous round for player 2 was Cl,CL, then Cl, Cl must be an NE in order for tit-for-tat to be an SPNE. Thus player 1 must be indifferent between CL and Dcl given player 2 plays Cl.

Player 2

Player 1	Cl	DCl
Cl	$4 + \delta \frac{4}{1-\delta}$, $4 + \delta \frac{4}{1-\delta}$	$0 + \delta \frac{5}{1-\delta^2}$, $6 + \delta \frac{6\delta}{1-\delta^2}$
Dcl	$5 + \delta \frac{\delta 5}{1-\delta^2}$, $0 + \delta \frac{6}{1-\delta^2}$	$1 + \frac{\delta}{1-\delta^2}$, $1 + \frac{\delta}{1-\delta^2}$

PS7, Ex. 6: Bertrand duopoly (infinitely repeated game)

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Exercise 2.13 in Gibbons (p. 135): Recall the static Bertrand duopoly model (with homogeneous products) from Problem 1.7: the firms name prices simultaneously; demand for firm i's product is $a - p_i$ if $p_i < p_j$, is 0 if $p_i > p_j$, and is $(a - p_i)/2$ if $p_i = p_j$; marginal costs are $c < a$. Consider the infinitely repeated game based on this stage game. Show that the firms can use trigger strategies (that switch forever to the stage-game Nash equilibrium after any deviation) to sustain the monopoly price level in a subgame-perfect Nash equilibrium if and only if $\delta \geq 1/2$.

PS7, Ex. 6: Bertrand duopoly (infinitely repeated game)

Show that the firms can use trigger strategies (that switch forever to the stage-game Nash equilibrium after any deviation) to sustain the monopoly price level in a subgame-perfect Nash equilibrium if and only if $\delta \geq 1/2$.

Information so far:

1. Players: Firm i, $i \in 1, 2$
2. Strategies: $S_i = \{p_i | p_i \in \mathbb{R}_+\}$
3. Payoffs:

$$\pi_i(p_i, p_j) = (price - cost) \cdot demand$$

$$= \begin{cases} (p_i - c)(a - p_i) & \text{if} \quad p_i < p_j \\ \frac{1}{2}(p_i - c)(a - p_i) & \text{if} \quad p_i = p_j \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

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Show that the firms can use trigger strategies (that switch forever to the stage-game Nash equilibrium after any deviation) to sustain the monopoly price level in a subgame-perfect Nash equilibrium if and only if $\delta \geq 1/2$.

Step a: Recall the stage game price level.

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$$= \begin{cases} (p_i - c)(a - p_i) & \text{if} \quad p_i < p_j \\ \frac{1}{2}(p_i - c)(a - p_i) & \text{if} \quad p_i = p_j \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

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a: Stage game NE: $p_1^* = p_2^* = c$

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Step b: Suggest a trigger strategy that can sustain the monopoly price level p^M.

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a: Stage game NE: $p_1^* = p_2^* = c$

b: Play p^M in $t = 0$ or if it was played in all previous rounds ("normal").

Play $p = c$ if there was a deviation in any previous round ("punishment").

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Show that the firms can use trigger strategies (that switch forever to the stage-game Nash equilibrium after any deviation) to sustain the monopoly price level in a subgame-perfect Nash equilibrium if and only if $\delta \geq 1/2$.

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Step c: Check that the trigger strategy (TS) is a NE in the "normal" phase.

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Step c: Check that the trigger strategy (TS) is a NE in the "normal" phase:

I.e. to split the monopoly market (LHS) vs the best deviation which is to slightly underbid, i.e. $p = p^M - \varepsilon \approx p^M$ (RHS).

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To split the monopoly market (LHS) vs the best deviation which is to slightly underbid, i.e. $p = p^M - \varepsilon \approx p^M$ (RHS):

$$\sum_{t=0}^{\infty} \frac{1}{2} \pi^M \delta^t \geq \pi^M + \sum_{t=1}^{\infty} \frac{1}{2} 0 \cdot \delta^t \Leftrightarrow$$

$$\frac{\frac{1}{2}\pi^M}{1-\delta} \geq \pi^M \Leftrightarrow$$

$$\frac{1}{2} \geq (1-\delta) \Leftrightarrow$$

$$1 \geq 2 - 2\delta \Leftrightarrow$$

$$\delta \geq \frac{1}{2}$$

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c: The TS is a NE in the "normal" phase for $\delta \geq \frac{1}{2}$

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Given that $p = c$ is a NE in the stage game, it must also be a NE in the "punishment" phase, i.e. it's a best response for both firms, given the other firm's price of $p = c$.

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Given that $p = c$ is a NE in the stage game, it must also be a NE in the "punishment" phase, i.e. it's a best response for both firms, given the other firm's price of $p = c$.

Thus, the trigger strategies gives a SPNE where the firms can act together as a monopolist if the they are sufficiently patient, i.e. for $\delta \geq \frac{1}{2}$.

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c: The TS is a NE in the "normal" phase for $\delta \geq \frac{1}{2}$

d: TS is SPNE for $\delta \geq \frac{1}{2}$