Microeconomics III: Problem Set 9[a]

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April 14 2020

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[a]Slides created for exercise class with reservation for possible errors.

Outline

PS9, Ex. 1 (A): Mixed-Strategy NE and Pure-Strategy BNE

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Consider this static game, where $k \in \mathbb{N}$:

	C	D
A	0, 2	2, 3
B	3, 1	k, 8

(a) For all possible values of k, find all Nash Equilibria (pure and mixed).

(b) Now assume that player 1 knows k, but player 2 only knows that $k = 1$ with probability $\frac{1}{2}$ and $k = 3$ with probability $\frac{1}{2}$. Find the Bayesian Nash Equilibrium of this game.

[Hints on the next slide. Try to independently write down the approach/criteria for a mixed-strategy NE and a pure-strategy BNE respectively.]

PS9, Ex. 1 (A): Mixed-Strategy NE and Pure-Strategy BNE

Consider this static game, where $k \in \mathbb{N}$:

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of k, find all Nash Equilibria (pure and mixed).

Hint: To find a mixed-strategy NE (MSNE):
Find the probabilities q for which Player 1 is indifferent, i.e. $u_1(A,q) = u_1(B,q)$ and the probabilities p for which Player 2 is indifferent, i.e. $u_2(C,p) = u_2(D,p)$.

(b) Now assume that player 1 knows k, but player 2 only knows that $k = 1$ with probability $\frac{1}{2}$ and $k = 3$ with probability $\frac{1}{2}$. Find the Bayesian Nash Equilibrium of this game.

Hint: Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for player 2 (the player with only one type, t_2). For each possible strategy $s_2(t_2)$:

1. Given the different types $t_{1,k} \in T_1 = \{t_{1,k=1}, t_{1,k=3}\}$, write up the best response of player 1: $s_1^*(t_{1,k}) \equiv BR_1\left(s_2(t_2)|t_{1,k}\right)$.
2. If it also holds that $s_2(t_2) = BR_2\left(s_1^*(t_{1,k})|t_2\right) \equiv s_2^*(t_2)$ then $\left(s_1^*(t_{1,k}), s_2^*(t_2)\right)$ is a BNE.

PS9, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where $k \in \mathbb{N}$:

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of k, find all Nash Equilibria (pure and mixed).

First, find all pure-strategy NE given k.

PS9, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where $k \in \mathbb{N}$:

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of k, find all Nash Equilibria (pure and mixed).

P2: C is strictly dominated by D, thus D is played in any NE, pure or mixed.

P1: For P2 playing D consider:

$k = 1$:

$k = 2$:

$k \geq 3$:

PS9, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where $k \in \mathbb{N}$:

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of k, find all Nash Equilibria (pure and mixed).

P2: C is strictly dominated by D, thus D is played in any NE, pure or mixed.

P1: For P2 playing D consider:

$k =$1: One PSNE: $\{(A, D)\}$

$k =$2: Two PSNE: $\{(A, D); (B, D)\}$

$k \geq$3: One PSNE: $\{(B, D)\}$

Then find all mixed-strategy NE given k.

PS9, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where $k \in \mathbb{N}$:

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of k, find all Nash Equilibria (pure and mixed).

P2: C is strictly dominated by D, thus D is played in any NE, pure or mixed.

P1: For P2 playing D consider:

$k = 1$: One PSNE: $\{(A, D)\}$

$k = 2$: Two PSNE: $\{(A, D); (B, D)\}$

$k \geq 3$: One PSNE: $\{(B, D)\}$

For P1 to mix, she has to be indifferent between A and B, thus we only need to look at:

$k = 2$:

PS9, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where $k \in \mathbb{N}$:

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of k, find all Nash Equilibria (pure and mixed).

P2: C is strictly dominated by D, thus D is played in any NE, pure or mixed.

P1: For P2 playing D consider:

$k = 1$: One PSNE: $\{(A, D)\}$

$k = 2$: Two PSNE: $\{(A, D); (B, D)\}$

$k \geq 3$: One PSNE: $\{(B, D)\}$

For P1 to mix, she has to be indifferent between A and B, thus we only need to look at:

$k = 2$: One MSNE: $\{(p \in (0, 1), q = 0)\}$

PS9, Ex. 1.b (A): Pure-Strategy BNE

Consider this static game, where $k \in \mathbb{N}$:

	C	D
A	0, 2	2, 3
B	3, 1	k, 8

(b) Now assume that player 1 knows k, but player 2 only knows that $k = 1$ with probability $\frac{1}{2}$ and $k = 3$ with probability $\frac{1}{2}$. Find the Bayesian Nash Equilibrium of this game.

[Hint for BNE on next slide.]

PS9, Ex. 1.b (A): Pure-Strategy BNE

Consider this static game, where $k \in \mathbb{N}$:

	C	D
A	0, 2	2, 3
B	3, 1	k, 8

(b) Now assume that player 1 knows k, but player 2 only knows that $k = 1$ with probability $\frac{1}{2}$ and $k = 3$ with probability $\frac{1}{2}$. Find the Bayesian Nash Equilibrium of this game.

Hint: Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for player 2 (the player with only one type, t_2). For each possible strategy $s_2(t_2)$:

Step 1: Given the different types $t_{1,k} \in T_1 = \{t_{1,k=1}, t_{1,k=3}\}$, write up the best response of player 1: $s_1^*(t_{1,k}) \equiv BR_1\left(s_2(t_2)|t_{1,k}\right)$.

Step 2: If it also holds that $s_2(t_2) = BR_2\left(s_1^*(t_{1,k})|t_2\right) \equiv s_2^*(t_2)$ then $\left(s_1^*(t_{1,k}), s_2^*(t_2)\right)$ is a BNE.

Consider this static game, where $k \in \mathbb{N}$:

	C	D
A	0, 2	2, 3
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(b) Now assume that player 1 knows k, but player 2 only knows that $k = 1$ with probability $\frac{1}{2}$ and $k = 3$ with probability $\frac{1}{2}$. Find the Bayesian Nash Equilibrium of this game.

Hint: Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for player 2 (the player with only one type, t_2). For each possible strategy $s_2(t_2)$:

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Step 2: If it also holds that $s_2(t_2) = BR_2\left(s_1^*(t_{1,k})|t_2\right) \equiv s_2^*(t_2)$ then $\left(s_1^*(t_{1,k}), s_2^*(t_2)\right)$ is a BNE.

As C is strictly dominated, player 2 only has the viable strategy $s_2(t_2) = D$:

1. Best response of player 1, $s_1^*(t_{1,k})$:

$$BR_1\left(D|t_{1,k}\right) = (s_1^*|t_{1,k=1}, s_1^*|t_{1,k=3}) = (A, B)$$

PS9, Ex. 1.b (A): Pure-Strategy BNE

Consider this static game, where $k \in \mathbb{N}$:

	C	D
A	0, 2	2, 3
B	3, 1	k, 8

(b) Now assume that player 1 knows k, but player 2 only knows that $k = 1$ with probability $\frac{1}{2}$ and $k = 3$ with probability $\frac{1}{2}$. Find the Bayesian Nash Equilibrium of this game.

Hint: Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for player 2 (the player with only one type, t_2). For each possible strategy $s_2(t_2)$:

Step 1: Given the different types $t_{1,k} \in T_1 = \{t_{1,k=1}, t_{1,k=3}\}$, write up the best response of player 1: $s_1^*(t_{1,k}) \equiv BR_1\left(s_2(t_2)|t_{1,k}\right)$.

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As C is strictly dominated, player 2 only has the viable strategy $s_2(t_2) = D$:

1. Best response of player 1, $s_1^*(t_{1,k})$:

$$BR_1\left(D|t_{1,k}\right) = (s_1^*|t_{1,k=1}, s_1^*|t_{1,k=3}) = (A, B)$$

2. As $D = BR_2\left((A, B)|t_2\right) \equiv s_2^*(t_2)$ we have a unique BNE:

$$\left((s_1^*|t_{1,k=1}, s_1^*|t_{1,k=3}), s_2^*(t_2)\right) = \{((A, B), D)\}$$

PS9, Ex. 2 (A): Mixed-Strategy BNE

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Consider the same set-up as exercise 3.4 in Gibbons, but now with the following bi-matrices for Game 1 and Game 2 respectively:

G1:

	L	R
U	1, 1	0, 0
D	0, 0	2, 0

G2:

	L	R
U	0, 0	0, 0
D	0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between L and R.

Exercise 3.4 in Gibbons (p. 169). Find all the pure-strategy Bayesian Nash equilibria in the following static Bayesian game:

a. *Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.*
b. ***Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not.***
c. *Player 1 chooses either U or D; player 2 simultaneously chooses either L or R.*
d. *Payoffs are given by the game drawn by nature.*

PS9, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:

	L	R
U	1, 1	0, 0
D	0, 0	2, 0

G2:

	L	R
U	0, 0	0, 0
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Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between L and R.

Step 1: **Find the player 1 strategies s_1' for which player 2 will want to mix.**

PS9, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:

	L	R
U	1, 1	0, 0
D	0, 0	2, 0

G2:

	L	R
U	0, 0	0, 0
D	0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between L and R.

Step 1: **Find the player 1 strategies s_1' for which player 2 will want to mix.**

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2: U is weekly dominated by D, thus, P1 will play D in G2 as long as P2 plays R with positive probability.

I.e. P1 either plays (U, D) or (D, D).

PS9, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

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	L	R
U	1, 1	0, 0
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Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between L and R.

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I.e. P1 either plays (U, D) or (D, D).

$BR_2((D, D)) = R$ but P2 is indifferent between L and R for $s_1' = (U, D)$.

PS9, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:

	L (q)	R (1-q)
U	1, 1	0, 0
D	0, 0	2, 0

G1:

	L (q)	R (1-q)
U	0, 0	0, 0
D	0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between L and R.

Step 1: Find the player 1 strategies s_1' for which player 2 will want to mix.

Step 2: **Find the values of q (the probability that player 2 plays L) such that player 1 will actually play s_1'.**

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2: U is weekly dominated by D, thus, P1 will play D in G2 as long as P2 plays R with positive probability.

I.e. P1 either plays (U, D) or (D, D).

$BR_2((D, D)) = R$ but P2 is indifferent between L and R for $s_1' = (U, D)$.

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1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U,U),(U,D),(D,D),(D,U)\}$$

G2: U is weekly dominated by D, thus, P1 will play D in G2 as long as P2 plays R with positive probability.

I.e. P1 either plays (U,D) or (D,D).

$BR_2((D,D)) = R$ but P2 is indifferent between L and R for $s_1' = (U,D)$.

2. P1 wants to play U in G1 if:

$$\mathbb{E}[u_1|U] \geq \mathbb{E}[u_1|D] \Leftrightarrow$$

$$q \geq 2(1-q) \Leftrightarrow 3q \geq 2 \Leftrightarrow q \geq \frac{2}{3}$$

PS9, Ex. 2 (A): Mixed-Strategy BNE

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G1:

	L (q)	R (1-q)
U	1, 1	0, 0
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G1:

	L (q)	R (1-q)
U	0, 0	0, 0
D	0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between L and R.

Step 1: Find the player 1 strategies s_1' for which player 2 will want to mix.

Step 2: Find the values of q (the probability that player 2 plays L) such that player 1 will actually play s_1'.

Step 3: **Write up the mixed-strategy BNE where P1 plays a pure strategy and P2 randomizes between *L* and *R*.**

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U,U),(U,D),(D,D),(D,U)\}$$

G2: U is weekly dominated by D, thus, P1 will play D in G2 as long as P2 plays R with positive probability.

I.e. P1 either plays (U,D) or (D,D).

$BR_2((D,D)) = R$ but P2 is indifferent between L and R for $s_1' = (U,D)$.

2. P1 wants to play U in G1 if:

$$\mathbb{E}[u_1|U] \geq \mathbb{E}[u_1|D] \Leftrightarrow$$

$$q \geq 2(1-q) \Leftrightarrow 3q \geq 2 \Leftrightarrow q \geq \frac{2}{3}$$

PS9, Ex. 2 (A): Mixed-Strategy BNE

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G1:

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Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between L and R.

Step 1: Find the player 1 strategies s_1' for which player 2 will want to mix.

Step 2: Find the values of q (the probability that player 2 plays L) such that player 1 will actually play s_1'.

Step 3: **Write up the mixed-strategy BNE where P1 plays a pure strategy and P2 randomizes between L and R.**

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2: U is weekly dominated by D, thus, P1 will play D in G2 as long as P2 plays R with positive probability.

I.e. P1 either plays (U, D) or (D, D).

$BR_2((D, D)) = R$ but P2 is indifferent between L and R for $s_1' = (U, D)$.

2. P1 wants to play U in G1 if:

$$\mathbb{E}[u_1|U] \geq \mathbb{E}[u_1|D] \Leftrightarrow$$

$$q \geq 2(1-q) \Leftrightarrow 3q \geq 2 \Leftrightarrow q \geq \frac{2}{3}$$

3. For q being the probability that P2 plays L, the mixed-strategy BNE is:

$$BNE' = \left\{ \left((U, D), q \geq \frac{2}{3} \right) \right\}$$

The expected highest and second-highest draw from a uniform distribution

The expected highest and second-highest draw from a uniform distribution

To find *seller's expected revenue* from a sealed bid auction (e.g. bidders simultaneously submit their bids in sealed envelopes without knowing the bids of others) with symmetric bidders with valuation drawn from a uniform distribution, there are two different approaches:

1. One approach is to derive each *bidder's expected payment* as a function of her valuation and then integrate this expression using the PDF to get the *ex-ante expected payment* of each bidder which can then be added together to find seller's expected revenue.
2. However, a more simple approach is to for N number of bidders to calculate the expected highest value (first-price sealed bid auction) or the expected second-highest value (second-price sealed bid auction). Plugging the value into the bid-function gives the seller's expected revenue.

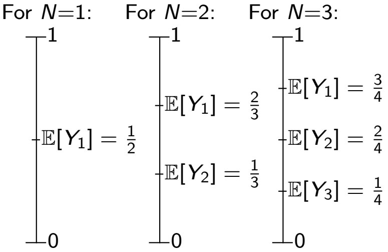
Deriving the optimal bid-function is a prerequisite for both approaches.

The expected highest and second-highest draw from a uniform distribution

First, let $X = x_1, x_2, ..., x_N$ be N independent and identically distributed (i.i.d.) draws from the ***standard* uniform distribution** $\boldsymbol{x \sim U(0, 1)}$. The highest draw Y_1 and the second-highest draw Y_2 of all N draws are expected to be:

$$\mathbb{E}[Y_1] = \frac{N}{N+1}, \quad \text{where } Y_1 = max(X)$$

$$\mathbb{E}[Y_2] = \frac{N-1}{N+1}, \quad \text{where } Y_2 = max(X \neq Y_1)$$



The expected highest and second-highest draw from a uniform distribution

First, let $X = x_1, x_2, ..., x_N$ be N independent and identically distributed (i.i.d.) draws from the *standard* uniform distribution $x \sim U(0,1)$. The highest draw Y_1 and the second-highest draw Y_2 of all N draws are expected to be:

$$\mathbb{E}[Y_1] = \frac{N}{N+1}, \quad \text{where } Y_1 = max(X)$$

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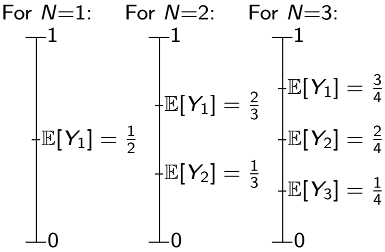

Generalized, let $X = x_1, x_2, ..., x_N$ be N independent and identically distributed (i.i.d.) draws from a uniform distribution $x \sim U(a, b)$. The highest draw Y_1 and the second-highest draw Y_2 of all N draws are expected to be:

$$\mathbb{E}[Y_1] = a + (b-a)\frac{N}{N+1}, \qquad \text{where } Y_1 = max(X)$$

$$\mathbb{E}[Y_2] = a + (b-a)\frac{N-1}{N+1}, \qquad \text{where } Y_2 = max(X \neq Y_1)$$

E.g. for $N = 1$, $\mathbb{E}[Y_1]$ simply collapses to the expression for the expected mean, μ:

$$\mathbb{E}[Y_1] = a + (b-a)\frac{N}{N+1} = a + (b-a)\frac{1}{1+1} = \frac{2a}{2} + \frac{b-a}{2} = \frac{a+b}{2} \equiv \mu$$

The expected highest and second-highest draw from a uniform distribution

Applied to auctions: Consider N number of bidders where each bidder i has the value v_i that is independently drawn from the same uniform distribution $v_i \sim U(a, b)$.

1st step: The highest value Y_1 and the second-highest value Y_2 for all N bidders are expected to be:

$$\mathbb{E}[Y_1] = a + (b-a)\frac{N}{N+1}, \qquad \text{where } Y_1 = max(V),\ V = v_1, v_2, ..., v_N$$

$$\mathbb{E}[Y_2] = a + (b-a)\frac{N-1}{N+1}, \qquad \text{where } Y_2 = max(V \neq Y_1)$$

2nd step: To calculate the seller's expected revenue, insert the expected highest value $\mathbb{E}[Y_1]$ (first-price sealed bid auction) or the expected second-highest value $\mathbb{E}[Y_2]$ (second-price sealed bid auction) in the derived bid-function.

The expected highest and second-highest draw from a uniform distribution

Proof: [only for those interested] Let $X = x_1, x_2, ..., x_N$ be N independent and identically distributed (i.i.d.) draws from a uniform distribution $x \sim U(a, b)$. Denote the highest draw $Y_1 = max(X)$. The cumulative distribution function (CDF) of Y_1 is:

$$
\begin{aligned}
G(x) &= \mathbb{P}[Y_1 \leq x] \\
&= \mathbb{P}[x_1 \leq x, x_1 \leq x, ..., x_N \leq x], && \text{since } Y_1 \text{ is the max of } X \\
&= \mathbb{P}[x_1 \leq x] \times \mathbb{P}[x_1 \leq x] \times ... \times \mathbb{P}[x_N \leq x], && \text{since draws are independent} \\
&= F(x) \times F(x) \times ... \times F(x) = (F(x))^N, && F(x) \text{ is the CDF of } x : F(x) = \frac{x-a}{b-a} \ (*)
\end{aligned}
$$

The first-derivative of the CDF gives the probability density function (PDF) of Y_1:

$$
\begin{aligned}
g(x) &= \frac{\delta G(x)}{\delta x} \\
&= F'(x) N (F(x))^{N-1} \\
&= f(x) N (F(x))^{N-1}, && f(x) \text{ is the PDF of } x : f(x) = \frac{1}{b-a} \ (**)
\end{aligned}
$$

The expectation to Y_1 is found by integrating x times the PDF of Y_1, $g(x)$:

$$
\begin{aligned}
\mathbb{E}[Y_1] &= \int_a^b x \cdot g(x) \ dx \\
&= \int_a^b x \cdot f(x) N (F(x))^{N-1} \ dx \\
&= \int_a^b x \cdot \tfrac{1}{b-a} N \left(\tfrac{x-a}{b-a}\right)^{N-1} \ dx, && \text{using } (**) \text{ and } (*)
\end{aligned}
$$

While the general solution isn't too obvious, the integral solves easily for $x \sim (0, 1)$:

$$\mathbb{E}[Y_1] = \int_0^1 x \tfrac{1}{1-0} N \left(\tfrac{x-0}{1-0}\right)^{N-1} dx = \int_0^1 x N x^{N-1} dx = \int_0^1 N x^N dx = \left[\tfrac{N}{N+1} x^{N+1}\right]_0^1 = \tfrac{N}{N+1}$$

PS9, Ex. 3: First- and second-price sealed bid auctions with two bidders

PS9, Ex. 3: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with

$$v_i \sim U(1, 3)$$

Thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$. Find c and d.

(b) Calculate the revenue to the seller.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

  i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

  ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

  iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

[Consider a uniform distribution $x \sim U(a, b)$. Try to write up the probability density function (PDF), cumulative distribution function (CDF), and the expectations to the highest value Y_1 and to the second-highest value Y_2 from N draws of x.]

PS9, Ex. 3: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with

$$v_i \sim U(1,3)$$

Thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

(b) Calculate the revenue to the seller.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for N draws from a uniform distribution $x \sim U(a,b)$:

PDF: Probability density function: $f(x) = \frac{1}{b-a}$

CDF: Cumulative distribution function: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a + (b-a)\frac{N}{N+1}$ where $Y_1 = \max(X)$, $X = x_1, x_2, ..., x_N$

$\mathbb{E}(Y_2) = a + (b-a)\frac{N-1}{N+1}$ where $Y_2 = \max(X \neq Y_1)$

PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with $v_i \sim U(1,3)$, thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

1st step: **Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i.**

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1},\ Y_1 = \max(X)$

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PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

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(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

1[st] step: **Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i:**

$$\mathbb{E}[u_i(b_i, v_i)] = \mathbb{P}(i \ wins|b_i)(v_i - b_i)$$

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

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PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

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1st step: Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i:

$$
\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\
&= \frac{b_i - d - c}{2c}(v_i - b_i)
\end{aligned}
$$

For N draws from $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

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PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

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1st step: Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i\ wins|b_i)(v_i - b_i) \\
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$$\mathbb{E}(Y_2) = a + (b\text{-}a)\frac{N-1}{N+1},\ Y_2 = \max(X \neq Y_1)$$

2nd step: **Take the FOC and SOC wrt. b_i.**

PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with $v_i \sim U(1,3)$, thus, the values are *private*.

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Results:

2nd: FOC: $\frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$

2nd step: **Take the FOC and SOC wrt. b_i.**

PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with $v_i \sim U(1,3)$, thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

1st step: Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i:

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\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins | b_i)(v_i - b_i) \\
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2nd: FOC: $\frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$

SOC: $-\frac{1}{c} = 0$

i.e. expected utility is concave in b_i.

PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

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$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins|b_i)(v_i - b_i) \\
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&= \frac{b_i - d - c}{2c}(v_i - b_i)
\end{aligned}$$

2nd step: Take the FOC and SOC wrt. b_i.

3rd step: **To find c^* and d^*, compare the best-response function $b_i(v_i)$ to $(*)$.**

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

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Results:

2nd: FOC: $\frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$

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PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

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(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

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$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins}|b_i)(v_i - b_i) \\
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&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\
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Results:

2nd: FOC: $\frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$
SOC: $-\frac{1}{c} = 0$
i.e. expected utility is concave in b_i.

3rd: From the FOC, the BR function is:

$$2b_i = v_i + d + c \Rightarrow$$

$$b_i(v_i) = \underbrace{\frac{1}{2}}_{c^*} v_1 + \underbrace{\frac{1}{2}(d + c)}_{d^*}$$

PS9, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with $v_i \sim U(1,3)$, thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

1st step: Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i\ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\
&= \frac{b_i - d - c}{2c}(v_i - b_i)
\end{aligned}$$

2nd step: Take the FOC and SOC wrt. b_i.

3rd step: To find c^* and d^*, compare the best-response function $b_i(v_i)$ to $(*)$.

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

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2nd: FOC: $\frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$
SOC: $-\frac{1}{c} = 0$
i.e. expected utility is concave in b_i.

3rd: From the FOC, the BR function is:

$$b_i(v_i) = \underbrace{\frac{1}{2}}_{c^*} v_1 + \underbrace{\frac{1}{2}(d + c)}_{d^*}$$

Inserting the first term in the second term, $d^* = \frac{1}{2}(d^* + c^*) = \frac{1}{2}(d^* + \frac{1}{2})$, which solves for $c^* = d^* = \frac{1}{2}$.

PS9, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1},\ Y_1 = \max(X)$

$\mathbb{E}(Y_2) = a + (b\text{-}a)\frac{N-1}{N+1},\ Y_2 = \max(X \neq Y_1)$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i-c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

PS9, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1^{st} step: **Calculate the expected highest value v_i for $N = 2$ draws from the uniform distribution $v_i \sim U(1,3)$.**

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1},\ Y_1 = \max(X)$$

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Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

PS9, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1st step: Calculate the expected highest value v_i for $N = 2$ draws from the uniform distribution $v_i \sim U(1,3)$:

$$\mathbb{E}[Y_1] = a + (b-a)\frac{N}{N+1}$$
$$= 1 + (3-1)\frac{2}{2+1}$$
$$= 1 + \frac{4}{3} = \frac{7}{3}$$

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

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Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \; wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

1st: $\mathbb{E}[Y_1] = \frac{7}{3}$

PS9, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1[st] step: Calculate the expected highest value v_i for $N = 2$ draws from the uniform distribution $v_i \sim U(1, 3)$:

$$\mathbb{E}[Y_1] = a + (b - a)\frac{N}{N+1}$$
$$= 1 + (3 - 1)\frac{2}{2+1}$$
$$= 1 + \frac{4}{3} = \frac{7}{3}$$

2[nd] step: **Insert the expected highest value $\mathbb{E}[Y_1]$ in the bid-function $(*)$ to find the seller's expected revenue.**

For N draws from $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

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$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$

$\mathbb{E}(Y_2) = a+(b\text{-}a)\frac{N-1}{N+1}, \; Y_2 = \max(X \neq Y_1)$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \; wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i-c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

1[st]: $\mathbb{E}[Y_1] = \frac{7}{3}$

(b) Calculate the revenue to the seller.

1st step: Calculate the expected highest value v_i for $N = 2$ draws from the uniform distribution $v_i \sim U(1, 3)$:

$$\mathbb{E}[Y_1] = a + (b - a)\frac{N}{N+1}$$
$$= 1 + (3 - 1)\frac{2}{2+1}$$
$$= 1 + \frac{4}{3} = \frac{7}{3}$$

2nd step: Insert the expected highest value $\mathbb{E}[Y_1]$ in the bid-function $(*)$ to find the seller's expected revenue:

$$\mathbb{E}[revenue] = b_i\left(\frac{7}{3}\right)$$
$$= c^*\frac{7}{3} + d^*$$
$$= \frac{1}{2}\cdot\frac{7}{3} + \frac{1}{2}, \qquad \text{using (3.a)}$$
$$= \frac{7}{6} + \frac{3}{6} = \frac{10}{6} = \frac{5}{3}$$

For N draws from $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

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(3.a) $c^* = d^* = \frac{1}{2}$

1st: $\mathbb{E}[Y_1] = \frac{7}{3}$

PS9, Ex. 3.c: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

   i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.
   ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
   iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

For N draws from $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$

$\mathbb{E}(Y_2) = a+(b\text{-}a)\frac{N-1}{N+1}, \; Y_2 = \max(X \neq Y_1)$

PS9, Ex. 3.c.i: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. **Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.**

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

For N draws from $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$

$\mathbb{E}(Y_2) = a + (b\text{-}a)\frac{N-1}{N+1}, \; Y_2 = \max(X \neq Y_1)$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

PS9, Ex. 3.c.ii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

For N draws from $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1},\ Y_1 = \max(X)$

$\mathbb{E}(Y_2) = a+(b\text{-}a)\frac{N-1}{N+1},\ Y_2 = \max(X \neq Y_1)$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) P1 wins: Payoff is independent of b_1 unless $b_1 < b_2$, in which case P1 no longer wins, thus, gets zero payoff.

P1 looses: Payoff is independent of b_1 unless $b_1 > b_2$, in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium.** Compare to the answer in (b).

For N draws from $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1}$, $Y_1 = \max(X)$

$\mathbb{E}(Y_2) = a+(b\text{-}a)\frac{N-1}{N+1}$, $Y_2 = \max(X \neq Y_1)$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

PS9, Ex. 3.c.iii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium.** Compare to the answer in (b).

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1},\ Y_1 = \max(X)$

$\mathbb{E}(Y_2) = a + (b\text{-}a)\frac{N-1}{N+1},\ Y_2 = \max(X \neq Y_1)$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) **Calculate the expected second-highest value v_i for $N = 2$ draws from the uniform distribution $v_i \sim U(1,3)$.**

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. **Compare to the answer in (b).**

For N draws from $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1},\ Y_1 = \max(X)$

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(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) Calculate the expected second-highest value v_i for $N = 2$ draws from the uniform distribution $v_i \sim U(1, 3)$:

$$\begin{aligned} \mathbb{E}[Y_2] &= a + (b-a)\frac{N-1}{N+1} \\ &= 1 + (3-1)\frac{1}{2+1} \\ &= 1 + \frac{2}{3} = \frac{5}{3} \end{aligned}$$

PS9, Ex. 3.c.iii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

For N draws from $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(k > x) = \frac{k-a}{b-a}$

$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$

$\mathbb{E}(Y_2) = a+(b\text{-}a)\frac{N-1}{N+1}, \; Y_2 = \max(X \neq Y_1)$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) Calculate the expected second-highest value v_i for $N = 2$ draws from the uniform distribution $v_i \sim U(1,3)$:

$$\begin{aligned} \mathbb{E}[Y_2] &= a + (b-a)\frac{N-1}{N+1} \\ &= 1 + (3-1)\frac{1}{2+1} \\ &= 1 + \frac{2}{3} = \frac{5}{3} \end{aligned}$$

Knowing from (ii) that each bidders bid their valuation:

$$\text{Seller's revenue} = \mathbb{E}[Y_2] = \frac{5}{3}$$

Thus, the outcome is the exact same as for the *first-price sealed bid auction*.

PS9, Ex. 4: First-price sealed bid auctions with three bidders

PS9, Ex. 4: First-price sealed bid auctions with three bidders

Consider the auction setting of the previous exercise. But now suppose that there are three identical bidders, $i = 1, 2, 3$, with values v_i where

$$v_i \sim U(1, 3)$$

and the values are independent, i.e. private. The auction is first-price sealed bid.

(a) Again, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

(c) (More difficult). Calculate the revenue to the seller.

PS9, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write bidder i's expected payoff in the proposed equilibrium.

PS9, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write bidder i's expected payoff in the proposed equilibrium:

$$\mathbb{E}[u_i(b_i, v_i)] = \mathbb{P}(i \ wins|b_i)(v_i - b_i)$$

PS9, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write i's expected payoff in eq.:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
\end{aligned}$$

PS9, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write i's expected payoff in eq.:

$$
\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins | b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
\end{aligned}
$$

Take the FOC and isolate $b_i^{**}(v_i)$.

PS9, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write i's expected payoff in eq.:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
\end{aligned}$$

$$\begin{aligned}
FOC: \quad 0 &= \frac{1}{2c}\left[2(b_i - d - c)(v_i - b_i) - (b_i - d - c)^2\right] \\
0 &= 2(v_i - b_i) - (b_i - d - c), && \text{assuming } b_i - d - c \neq 0 \\
b_i^{**}(v_i) &= \frac{2}{3}v_i + \frac{1}{3}(c + d)
\end{aligned}$$

PS9, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write i's expected payoff in eq.:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
\end{aligned}$$

$$FOC: \quad 0 = \frac{1}{2c}[2(b_i - d - c)(v_i - b_i) - (b_i - d - c)^2]$$

$$0 = 2(v_i - b_i) - (b_i - d - c), \qquad \text{assuming } b_i - d - c \neq 0$$

$$b_i^{**}(v_i) = \underbrace{\frac{2}{3}}_{c^* = \frac{2}{3}} v_i + \underbrace{\frac{1}{3}(c + d)}_{d^* = \frac{1}{3}\left(\frac{2}{3} + d^*\right) \Rightarrow d^* = \frac{1}{3}} = \underline{\underline{\frac{2}{3}v_i + \frac{1}{3}}} \qquad Q.E.D.$$

PS9, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

For N draws from $x \sim U(a, b)$:

$$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$$

BNE found:

(3.a) $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$ for $i \in 1, 2, 3$

(3.b) $\mathbb{E}[revenue] = \frac{5}{3}$

(4.a) $b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3}$ for $i \in 1, 2, 3$

PS9, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading $\left(\frac{2}{3} > \frac{1}{2}\right)$ and therefore a *higher* revenue for the seller.

For N draws from $x \sim U(a, b)$:

$$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$$

BNE found:

(3.a) $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$ for $i \in 1, 2, 3$

(3.b) $\mathbb{E}[revenue] = \frac{5}{3}$

(4.a) $b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3}$ for $i \in 1, 2, 3$

PS9, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading $\left(\frac{2}{3} > \frac{1}{2}\right)$ and therefore a *higher* revenue for the seller.

Looking at the bid strategies, we can confirm this, as each bidder submits higher bids with more opponents:

$$\begin{aligned} b_i^{**} > b_i^{*} &\Leftrightarrow \\ \frac{2}{3}v_i + \frac{1}{3} > \frac{1}{2}v_i + \frac{1}{2} &\Leftrightarrow \\ \frac{1}{6}v_i > \frac{1}{6} &\Leftrightarrow \\ v_i > 1 & \end{aligned}$$

I.e. except for the rare case where all players have the valuation $v = 1$, the seller's revenue is strictly higher with three players than with two players.

For N draws from $x \sim U(a, b)$:

$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1}$, $Y_1 = \max(X)$

BNE found:

(3.a) $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$ for $i \in 1, 2, 3$

(3.b) $\mathbb{E}[revenue] = \frac{5}{3}$

(4.a) $b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3}$ for $i \in 1, 2, 3$

PS9, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading $\left(\frac{2}{3} > \frac{1}{2}\right)$ and therefore a *higher* revenue for the seller.

Looking at the bid strategies, we can confirm this, as each bidder submits higher bids with more opponents:

$$b_i^{**} > b_i^* \Leftrightarrow$$

$$\frac{2}{3}v_i + \frac{1}{3} > \frac{1}{2}v_i + \frac{1}{2} \Leftrightarrow$$

$$\frac{1}{6}v_i > \frac{1}{6} \Leftrightarrow$$

$$v_i > 1$$

I.e. except for the rare case where all players have the valuation $v = 1$, the seller's revenue is strictly higher with three players than with two players.

Furthermore, the expected highest value of N draws is increasing in N.

For N draws from $x \sim U(a, b)$:

$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$

BNE found:

(3.a) $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$ for $i \in 1, 2, 3$

(3.b) $\mathbb{E}[revenue] = \frac{5}{3}$

(4.a) $b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3}$ for $i \in 1, 2, 3$

PS9, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

For N draws from $x \sim U(a, b)$:

$$\mathbb{E}(Y_1) = a + (b\text{-}a)\frac{N}{N+1},\ Y_1 = \max(X)$$

Results so far:

$$(*)\quad b_i(v_i) = cv_i + d$$

$$(3.a)\quad c^* = d^* = \tfrac{1}{2}$$

$$(3.b)\quad \mathbb{E}[revenue] = \tfrac{5}{3}$$

$$(4.a)\quad c^* = \tfrac{2}{3},\ d^* = \tfrac{1}{2}$$

PS9, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1[st] step: **Calculate the expected highest value v_i for $N = 3$ draws from the uniform distribution $v_i \sim U(1,3)$.**

For N draws from $x \sim U(a,b)$:

$$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

(3.a) $c^* = d^* = \frac{1}{2}$

(3.b) $\mathbb{E}[revenue] = \frac{5}{3}$

(4.a) $c^* = \frac{2}{3}, \; d^* = \frac{1}{2}$

PS9, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected highest value v_i for $N = 3$ draws from the uniform distribution $v_i \sim U(1,3)$:

$$\mathbb{E}[Y_1] = a + (b-a)\frac{N}{N+1}$$
$$= 1 + (3-1)\frac{3}{3+1}$$
$$= 1 + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}$$

2nd step: **Insert the expected highest value $\mathbb{E}[Y_1]$ in the bid-function $(*)$ to find the seller's expected revenue.**

For N draws from $x \sim U(a,b)$:

$\mathbb{E}(Y_1) = a{+}(b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

(3.a) $c^* = d^* = \frac{1}{2}$

(3.b) $\mathbb{E}[revenue] = \frac{5}{3}$

(4.a) $c^* = \frac{2}{3}, \; d^* = \frac{1}{2}$

1st: $\mathbb{E}[Y_1] = \frac{5}{2}$

PS9, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected highest value v_i for $N = 3$ draws from the uniform distribution $v_i \sim U(1, 3)$:

$$\begin{aligned}\mathbb{E}[Y_1] &= a + (b-a)\frac{N}{N+1} \\ &= 1 + (3-1)\frac{3}{3+1} \\ &= 1 + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}\end{aligned}$$

2nd step: Insert the expected highest value $\mathbb{E}[Y_1]$ in the bid-function $(*)$ to find the seller's expected revenue.

3rd step: **Does this live up to your expectation in ex. (4.b)?**

For N draws from $x \sim U(a, b)$:

$\mathbb{E}(Y_1) = a+(b\text{-}a)\frac{N}{N+1}, \ Y_1 = \max(X)$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

(3.a) $c^* = d^* = \frac{1}{2}$

(3.b) $\mathbb{E}[revenue] = \frac{5}{3}$

(4.a) $c^* = \frac{2}{3}, \ d^* = \frac{1}{2}$

1st: $\mathbb{E}[Y_1] = \frac{5}{2}$

2nd: The seller's expected revenue:

$$\begin{aligned}\mathbb{E}[revenue] &= b_i\left(\frac{5}{2}\right) \\ &= c^*\frac{5}{2} + d^* \\ &= \frac{2}{3}\cdot\frac{5}{2} + \frac{1}{3}, \qquad \text{using (4.a)} \\ &= \frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2 > \frac{5}{3}\end{aligned}$$

PS9, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected highest value v_i for $N = 3$ draws from the uniform distribution $v_i \sim U(1, 3)$:

$$\mathbb{E}[Y_1] = a + (b - a)\frac{N}{N+1} = 1 + (3-1)\frac{3}{3+1} = 1 + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}$$

2nd step: Insert the expected highest value $\mathbb{E}[Y_1]$ in the bid-function $(*)$ to find the seller's expected revenue.

3rd step: Does this live up to your expectation in ex. (4.b)?

For N draws from $x \sim U(a, b)$:

$\mathbb{E}(Y_1) = a{+}(b\text{-}a)\frac{N}{N+1}, \; Y_1 = \max(X)$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

(3.a) $c^* = d^* = \frac{1}{2}$

(3.b) $\mathbb{E}[revenue] = \frac{5}{3}$

(4.a) $c^* = \frac{2}{3}, \; d^* = \frac{1}{2}$

1st: $\mathbb{E}[Y_1] = \frac{5}{2}$

2nd: The seller's expected revenue:

$$\begin{aligned}\mathbb{E}[revenue] &= b_i\left(\frac{5}{2}\right) \\ &= c^*\frac{5}{2} + d^* \\ &= \frac{2}{3}\cdot\frac{5}{2} + \frac{1}{3}, \qquad \text{using (4.a)} \\ &= \frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2 > \frac{5}{3}\end{aligned}$$

3rd: The expected revenue is higher as more players increases competition and the expected highest valuation.

PS9, Ex. 5: Winner's Curse

PS9, Ex. 5: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(a) Are the bidders' values private or common?

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

PS9, Ex. 5.a: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(a) Are the bidders' values private or common?

Though the reports investigating the values are private, the bidders' actual values are ***common*** since they are identical, i.e. $v_1 = v_2 = v$.

PS9, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Write up Bayes' rule.

PS9, Ex. 5.b: Winner's Curse

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The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

PS9, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: Use Bayes' rule and the given probabilities to write up the probability that the value of the oil-field is high after having received the report $r_1 = H$.

PS9, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H]} = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]}$$

$$= \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

PS9, Ex. 5.b: Winner's Curse

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H]} = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]}$$

$$= \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

Step 3: Use ($\star$) to write up the expected value of the oil-field after receiving the report $r_1 = H$ where the profits can be either high v_H or low v_L.

PS9, Ex. 5.b: Winner's Curse

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The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

Step 3: Use $(\star)$ to write up the expected value of the oil-field after receiving the report $r_1 = H$ where the profits can be either high v_H or low v_L:

$$\mathbb{E}[v|r_1 = H] = \mathbb{P}[H|r_1 = H]v_H + \mathbb{P}[L|r_1 = H]v_L$$

PS9, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

Step 3: Use $(\star)$ to write up the expected value of the oil-field after receiving the report $r_1 = H$ where the profits can be either high v_H or low v_L:

$$\begin{aligned}\mathbb{E}[v|r_1 = H] &= \mathbb{P}[H|r_1 = H]v_H + \mathbb{P}[L|r_1 = H]v_L \\ &= \mathbb{P}[H|r_1 = H]v_H + (1 - \mathbb{P}[H|r_1 = H])\, v_L, \text{ using } (\star) \\ &= pv_H + (1-p)v_L\end{aligned}$$

PS9, Ex. 5.c: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

PS9, Ex. 5.c: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: Write up the probability that the value of the oil-field is H given company 1 receives a report $r_1 = H$ and wins the auction.

PS9, Ex. 5.c: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L\text{)}$$

PS9, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\begin{aligned}
\mathbb{P}[H|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L\text{)} \\
&= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L]} \\
&= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)
\end{aligned}$$

PS9, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\begin{aligned}
\mathbb{P}[H|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L\text{)} \\
&= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L]} \\
&= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)
\end{aligned}$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction.

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L\text{)}$$

$$= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L]}$$

$$= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = L|L] \times \mathbb{P}[L]}$$

$$= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]}$$

$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\mathbb{E}[v|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge win]v_H + \mathbb{P}[L|r_1 = L \wedge win]v_L$$

PS9, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L\text{)}$$

$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge win]v_H + \mathbb{P}[L|r_1 = R \wedge win]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge win]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\, v_L\end{aligned}$$

Now, insert $(\star\star)$ above and compare to the result in (3.b).

PS9, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L)$$

$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge win]v_H + \mathbb{P}[L|r_1 = R \wedge win]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge win]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\,v_L \qquad \text{insert } (\star\star) \\ &= \frac{1}{2}v_H + \frac{1}{2}v_L < \underbrace{pv_H + (1-p)v_L}_{\mathbb{E}[v|r_1 = H],\ \text{using (b)}} \qquad \text{since } p > \frac{1}{2}\end{aligned}$$

Step 3: Looking at the inequality above, explain the difference between (b) and (c).

PS9, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L\text{)}$$

$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge win]v_H + \mathbb{P}[L|r_1 = R \wedge win]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge win]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\, v_L \quad \text{insert } (\star\star) \\ &= \frac{1}{2}v_H + \frac{1}{2}v_L < \underbrace{pv_H + (1-p)v_L}_{\mathbb{E}[v|r_1 = H],\ \text{using (b)}} \quad \text{since } p > \frac{1}{2}\end{aligned}$$

Step 3: Looking at the inequality above, explain the difference between (b) and (c). *In other words, why is company 1 less certain that the value is H after they actually win the auction?*

PS9, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L)$$

$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge win]v_H + \mathbb{P}[L|r_1 = R \wedge win]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge win]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\, v_L \quad \text{insert } (\star\star) \\ &= \frac{1}{2}v_H + \frac{1}{2}v_L < \underbrace{pv_H + (1-p)v_L}_{\mathbb{E}[v|r_1 = H],\ \text{using (b)}} \quad \text{since } p > \frac{1}{2}\end{aligned}$$

Step 3: This is an example of The Winner's Curse: The equally trustworthy reports of the two companies cancel each other out. Since the valuations of the auctioned object are correlated, you are likely to win the object when you overestimate the value.

PS9, Ex. 3': Solving 3.b and 3.c through integration (optional)

PS9, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

Standard results for $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

($*$) $b_i(v_i) = cv_i + d$

($**$) $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

(b) Calculate the revenue to the seller.

1^{st} step: **Calculate the expected payment of bidder *i* with valuation v_i.**

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i-c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

PS9, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1st step: **Calculate the expected payment of bidder i with valuation v_i:**

$$m_i(v_i) = \mathbb{P}(i \ wins|v_i)b_i(v_i)$$

Standard results for $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i-c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

PS9, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1^{st} step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i\ wins|v_i)b_i(v_i) \\
&= \frac{cv_i - c}{2c}(cv_i + d), \text{ using } (*), (**) \\
&= \frac{v_i - 1}{2}(cv_i + d) \\
&= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \text{ using (3.a)} \\
&= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\
&= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4}
\end{aligned}
$$

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i-c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

PS9, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i \ wins|v_i)b_i(v_i) \\
&= \frac{cv_i - c}{2c}(cv_i + d), \ \text{using } (*), (**) \\
&= \frac{v_i - 1}{2}(cv_i + d) \\
&= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \ \text{using (3.a)} \\
&= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\
&= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4}
\end{aligned}
$$

2nd step: **Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.**

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i-c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

PS9, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i\ wins|v_i) b_i(v_i) \\ &= \frac{cv_i - c}{2c}(cv_i + d), \text{ using } (*), (**) \\ &= \frac{v_i - 1}{2}(cv_i + d) \\ &= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \text{ using (3.a)} \\ &= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\ &= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4} \end{aligned}$$

2nd step: **Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.**

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

PS9, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned}
m_i(v_i) &= \mathbb{P}(i\ wins|v_i)b_i(v_i) \\
&= \frac{cv_i - c}{2c}(cv_i + d),\ \text{using } (*), (**) \\
&= \frac{v_i - 1}{2}(cv_i + d) \\
&= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right),\ \text{using (3.a)} \\
&= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\
&= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4}
\end{aligned}$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\begin{aligned}
\mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i)f_i(v_i)dv_i \\
&= \int_1^3 \frac{v_i^2-1}{4} \cdot \frac{1}{3-1} dv_i \\
&= \frac{1}{8}\int_1^3 v_i^2 - 1 dv_i \\
&= \frac{1}{8}\left[\frac{1}{3}v_i^3 - v_i\right]_1^3 \\
&= \frac{1}{8}\left(\frac{3^3}{3} - 3 - \frac{1^3}{3} + 1\right) = \frac{5}{6}
\end{aligned}$$

PS9, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \ wins|v_i) b_i(v_i) \\ &= \frac{cv_i - c}{2c}(cv_i + d), \text{ using } (*), (**) \\ &= \frac{v_i - 1}{2}(cv_i + d) \\ &= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \text{ using (3.a)} \\ &= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\ &= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4} \end{aligned}$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

3rd step: **Write up the expected revenue to the seller.**

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \ wins|v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\begin{aligned} \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\ &= \int_1^3 \frac{v_i^2 - 1}{4} \cdot \frac{1}{3-1} dv_i \\ &= \frac{1}{8}\int_1^3 v_i^2 - 1 dv_i \\ &= \frac{1}{8}\left[\frac{1}{3}v_i^3 - v_i\right]_1^3 \\ &= \frac{1}{8}\left(\frac{3^3}{3} - 3 - \frac{1^3}{3} + 1\right) = \frac{5}{6} \end{aligned}$$

PS9, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i \ wins|v_i)b_i(v_i) \\
&= \frac{cv_i - c}{2c}(cv_i + d), \text{ using } (*), (**) \\
&= \frac{v_i - 1}{2}(cv_i + d) \\
&= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \text{ using (3.a)} \\
&= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\
&= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4}
\end{aligned}
$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

3rd step: The expected revenue to the seller is the ex-ante expected payment of both bidders:

$$\text{Seller's revenue} = \mathbb{E}[m_1(v_1)] + \mathbb{E}[m_2(v_2)] = \frac{5}{3}$$

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$
\begin{aligned}
\mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i)f_i(v_i)dv_i \\
&= \int_1^3 \frac{v_i^2-1}{4} \cdot \frac{1}{3-1} dv_i \\
&= \frac{1}{8}\int_1^3 v_i^2 - 1 dv_i \\
&= \frac{1}{8}\left[\frac{1}{3}v_i^3 - v_i\right]_1^3 \\
&= \frac{1}{8}\left(\frac{3^3}{3} - 3 - \frac{1^3}{3} + 1\right) = \frac{5}{6}
\end{aligned}
$$

PS9, Ex. 3.c': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

  i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.
  ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
  iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

PS9, Ex. 3.c.i': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.
ii. **Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.**
iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

PS9, Ex. 3.c.ii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) P1 wins: Payoff is independent of b_1 unless $b_1 < b_2$, in which case P1 no longer wins, thus, gets zero payoff.

P1 looses: Payoff is independent of b_1 unless $b_1 > b_2$, in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) **Calculate player i's expected payment in the BNE.**

PS9, Ex. 3.c.iii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.
ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for $x \sim U(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) **Calculate player *i*'s expected payment in the BNE:**

$$m_i(v_i) = \mathbb{P}(i\ wins|v_i) \cdot \mathbb{E}[b_j^*(v_j)|b_j^*(v_j) < b_i^*(v_i)]$$

PS9, Ex. 3.c.iii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

  i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.
  ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
  iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) Player i's expected payment in BNE:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \ wins|v_i) \cdot \mathbb{E}[b_j^*(v_j)|b_j^*(v_j) < b_i^*(v_i)] \\ &= \mathbb{P}(v_i > v_j) \cdot \mathbb{E}[v_j|v_j < v_i] \\ &= \frac{v_i - 1}{3 - 1} \cdot \frac{1 + v_i}{2}, \text{ using CDF and Mean} \\ &= \frac{v_i + v_i^2 - 1^2 - v_i}{2^2} = \frac{v_i^2 - 1}{4} \end{aligned}$$

PS9, Ex. 3.c.iii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for $x \sim U(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

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(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

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(iii) Player i's expected payment in BNE:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \text{ } wins|v_i) \cdot \mathbb{E}[b_j^*(v_j)|b_j^*(v_j) < b_i^*(v_i)] \\ &= \mathbb{P}(v_i > v_j) \cdot \mathbb{E}[v_j|v_j < v_i] \\ &= \frac{v_i - 1}{3-1} \cdot \frac{1+v_i}{2}, \text{ using CDF and Mean} \\ &= \frac{v_i + v_i^2 - 1^2 - v_i}{2^2} = \frac{v_i^2 - 1}{4} \end{aligned}$$

As this is the same as in (3.b), we know:

Ex-ante expected payment $= \mathbb{E}[m_i(v_i)] = \frac{5}{6}$

Seller's revenue $= 2 \cdot \mathbb{E}[m_i(v_i)] = \frac{5}{3}$

Thus, the outcome is the exact same as for the *first-price sealed bid auction*.

PS9, Ex. 4’: Solving 4.c through integration (optional)

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: **Calculate the expected payment of bidder *i* with valuation v_i.**

Results so far:

($*$) $b_i(v_i) = cv_i + d$

($\dagger$) $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: **Calculate the expected payment of bidder *i* with valuation v_i:**

$$m_i(v_i) = \mathbb{P}(i \ wins|v_i) b_i(v_i)$$

Results so far:

($*$) $b_i(v_i) = cv_i + d$

($\dagger$) $\mathbb{P}(i \ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3}, \ d^* = \frac{1}{2}$

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$m_i(v_i) = \mathbb{P}(i\ wins|v_i) b_i(v_i)$$
$$= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ using } (\dagger)$$
$$= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ using } (*)$$
$$= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right), \text{ using } (4.a)$$
$$= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)$$

Results so far:

($*$) $b_i(v_i) = cv_i + d$

($\dagger$) $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i\ wins|v_i) b_i(v_i) \\ &= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ using } (\dagger) \\ &= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ using } (*) \\ &= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right), \text{ using } (4.a) \\ &= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \end{aligned}$$

2nd step: **Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.**

Results so far:

($*$) $b_i(v_i) = cv_i + d$

($\dagger$) $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i \; wins|v_i) b_i(v_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ using } (\dagger) \\
&= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ using } (*) \\
&= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right), \text{ using } (4.a) \\
&= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)
\end{aligned}
$$

2nd step: **Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.**

Results so far:

($*$) $b_i(v_i) = cv_i + d$

($\dagger$) $\mathbb{P}(i \; wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3}, \; d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$m_i(v_i) = \mathbb{P}(i\ wins|v_i)b_i(v_i)$$

$$= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i),\ \text{using } (\dagger)$$

$$= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d),\ \text{using } (*)$$

$$= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right),\ \text{using } (4.a)$$

$$= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Why is the ex-ante expected payment lower than in exercise 3.b?

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

$$= \int_1^3 \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \cdot \frac{1}{3-1} dv_i$$

$$= \frac{1}{24}\left[\frac{2}{4}v_i^4 - \frac{3}{3}v_i^3 + v_i\right]_1^3$$

$$= \frac{1}{24}\left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6}$$

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i\ wins|v_i) b_i(v_i) \\ &= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ using } (\dagger) \\ &= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ using } (*) \\ &= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3} v_i + \frac{1}{3}\right), \text{ using } (4.a) \\ &= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \end{aligned}$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\begin{aligned} \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\ &= \int_1^3 \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \cdot \frac{1}{3-1} dv_i \\ &= \frac{1}{24} \left[\frac{2}{4} v_i^4 - \frac{3}{3} v_i^3 + v_i\right]_1^3 \\ &= \frac{1}{24} \left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6} \end{aligned}$$

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i\ wins|v_i) b_i(v_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ using } (\dagger) \\
&= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ using } (*) \\
&= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3} v_i + \frac{1}{3}\right), \text{ using } (4.a) \\
&= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)
\end{aligned}
$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3rd step: **Calculate the seller's revenue and compare to exercise (3.b).**

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$
\begin{aligned}
\mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\
&= \int_1^3 \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \cdot \frac{1}{3-1} dv_i \\
&= \frac{1}{24} \left[\frac{2}{4} v_i^4 - \frac{3}{3} v_i^3 + v_i\right]_1^3 \\
&= \frac{1}{24} \left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6}
\end{aligned}
$$

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i\ wins|v_i)b_i(v_i) \\
&= \left(\frac{b_i-d-c}{2c}\right)^2 b_i(v_i), \text{ using } (\dagger) \\
&= \left(\frac{cv_i-c}{2c}\right)^2 (cv_i+d), \text{ using } (*) \\
&= \left(\frac{v_i-1}{2}\right)^2 \left(\frac{2}{3}v_i+\frac{1}{3}\right), \text{ using } (4.a) \\
&= \left(\frac{2v_i^3-3v_i^2+1}{12}\right)
\end{aligned}
$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3rd step: Calculate the seller's revenue and compare to exercise (3.b).

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i-d-c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$
\begin{aligned}
\mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i)f_i(v_i)dv_i \\
&= \int_1^3 \left(\frac{2v_i^3-3v_i^2+1}{12}\right)\cdot\frac{1}{3-1}dv_i \\
&= \frac{1}{24}\left[\frac{2}{4}v_i^4-\frac{3}{3}v_i^3+v_i\right]_1^3 \\
&= \frac{1}{24}\left(\frac{33}{2}-\frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6}
\end{aligned}
$$

3rd: $Revenue = 3\cdot\mathbb{E}[m_i(v_i)] = 2 > \frac{5}{3}$

Why is seller's revenue higher than in exercise 3.b?

PS9, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i\ wins|v_i)b_i(v_i) \\ &= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ using } (\dagger) \\ &= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ using } (*) \\ &= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right), \text{ using } (4.a) \\ &= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \end{aligned}$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3rd step: Calculate the seller's revenue and compare to exercise (3.b).

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\begin{aligned} \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\ &= \int_1^3 \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \cdot \frac{1}{3-1} dv_i \\ &= \frac{1}{24}\left[\frac{2}{4}v_i^4 - \frac{3}{3}v_i^3 + v_i\right]_1^3 \\ &= \frac{1}{24}\left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6} \end{aligned}$$

3rd: $Revenue = 3 \cdot \mathbb{E}[m_i(v_i)] = 2 > \frac{5}{3}$

The seller can expect higher revenue as more players increases competition and the chance of one having high valuation.